Modified Types of Triple Effect Domination

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Abstract
Let $G = (V, E)$ be a finite, simple and undirected graph without isolated vertices. A sub set $D \subseteq V$ is a triple effect dominating set, if every vertex in D dominates exactly three vertices of $V - D$. Triple effect domination number $\gamma_{te}(G)$ is the minimum cardinality over all triple effect dominating sets in G. A subset D^{-1} of *V-D* is an inverse triple effect dominating set if every $v \in D^{-1}$ dominates exactly three vertices of $V - D^{-1}$. The inverse triple effect domination number $\gamma_{te}^{-1}(G)$ is the minimum cardinality over all inverse triple effect dominating sets in G. In this papers, total, independent, co-independent, connected and doubly connected triple effect domination are introduced with their inverse as a modified of the triple effect domination. Several properties and bounds are given and proved. Then, these modified dominations are applied on some graphs.

Keywords: Dominating set, triple effect domination, inverse triple effect domination.

Mathematics Subject Classifications: 05C12, 05C19

Introduction

Let $G = (V, E)$ be a graph with order n and size m, such that n the number of all vertices in G and m the number of all edges in G. The open neighborhood of v is $N(t) = \{r \in V \mid vr \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of any vertex v in G is the number of edges incident on v and denoted by $\deg(v)$. If $\deg(v) = 0$, then v is said isolated vertex. $\Delta(G)$ is the maximum degree in G and $\delta(G)$ is the minimum degree in G. The induced subgraph of a subset vertex N of V and the edges between them is $G[N]$. The complement of a simple graph G denoted by $\bar{G}$, it is a graph with the same vertices of G and there is an edge between any two vertices in $\bar{G}$ if and only if there is no edge in G between them. See [22] for theoretic terminology and basic concepts of graphs. In G, a set D of V is said a dominating set if every vertex out it, is adjacent to one vertex or more of it. For a detailed survey of domination, one can see [17,19, 23, 24]. Several papers studied different types of domination such as [1-10, 14-16, 18, 20, 21, 25-33]. In previous papers [11-13] we study new model of domination called triple effect domination and introduced several theorems and properties. Also, we defined the inverse triple effect domination and discussed its properties and bounds. In this paper, the triple effect domination is modified by adding new conditions on D or its complement set. Total triple effect domination, independent triple effect domination, co-independent triple effect domination, connected triple effect domination, doubly connected triple effect domination and their inverse models are introduced here and applied with several bounds and properties.

2. Total triple effect domination

Total triple effect domination and the inverse total triple effect domination are defined here. Some properties and bounds are discussed and applied on some known graphs.

Definition 2.1: Let G be a graph, a set D of $V(G)$ is a total triple effect dominating set if D is a triple effect dominating set and $G[D]$ has no isolated vertex.

Definition 2.2: A total triple effect dominating set is minimal if it has no proper total triple effect dominating subset. D is minimum if it's the cardinality is smallest overall total triple effect dominating sets in G.

Definition 2.3: The total triple effect domination number denoted by $\gamma_{te}^{t}(G)$ is the cardinality of the minimum total triple effect dominating set in G. Such set is referred as γ_{te}^{t}-set.

Definition 2.4: Let G be a graph with γ_{te}^{t}-set D, a subset $D^{-1} \subseteq V - D$ is an inverse total triple effect dominating set with respect to D, if $D^{-1}$is a total triple effect dominating set.

Definition 2.5: An inverse total triple effect dominating set D^{-1} is minimal if it has no proper inverse total triple effect dominating subset. D^{-1} is minimum if its cardinality is smallest overall inverse total triple effect dominating sets in G.

Definition 2.6: The inverse total triple effect domination number denoted by $\gamma_{te}^{-t}(G)$ is the cardinality of the minimum inverse total triple effect dominating set in G. Such set is referred as γ_{te}^{-t}-set. For example see figure 1.

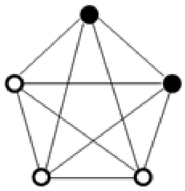

Figure 1:Total triple effect dominating set of K_5.

Remark 2.7: For any graph with total triple effect dominating set. Then $|V(G)| \geq 5$.
Remark 2.8: For any disconnected graph G with a total triple effect dominating set. Then $|V(H)| \geq 5$ for all component H of G.
Remark 2.9: For any graph with an inverse total triple effect dominating set. Then$|V(G)| \geq 5$.

Remark 2.10: Let G be a graph contains a pendant vertex. If G has a total triple effect domination, then G has no inverse total triple effect domination.

Theorem 2.11: Let $G(n, m)$ be a graph having total triple effect domination, then:

$$2 \leq \gamma_{te}^{t}(G) \leq n-3$$

Proof: Let D be a$\gamma_{te}^{t} - set$ of G then :
Case 1: Since $G[D]$ has no isolated vertex, then D has at least two adjacent vertices. Therefore, in general $\gamma_{te}^{t}(G) = |D| \geq 2$.
Case 2: Since $V - D$ has at least three vertices such that all the other $n-3$ vertices dominate these three vertices. Hence, $\gamma_{te}^{t}(G) = |D| \leq n-3$.

Remark 2.12: Let $G(n, m)$ be a graph having inverse total triple effect domination, then: $2 \leq \gamma_{te}^{-t}(G) \leq n-3$.

Theorem 2.13: Let $G(n, m), (n > 4)$ be any graph having total triple effect domination, then:

$$3\gamma_{te}^{t}(G) + \left\lceil \frac{\gamma_{te}^{t}(G)}{2} \right\rceil \leq m \leq \binom{n}{2} + \left(\gamma_{te}^{t}(G)\right)^2 + (3-n)\gamma_{te}^{t}(G)$$

Proof: Let D be a$\gamma_{te}^{t} - set$ of G, then:
Case 1: Let $G[V-D]$ be a null graph, and G has as few edges as possible.
Now, by the definition of the total triple effect domination, for every $v \in D$, then $\deg(v) = 4$ at least.Where v is adjacent with one vertex at least of D and dominates three vertices from $V - D$. Therefore, the number of edges between D and $V - D$ equal $m_1 = 3|D| = 3\gamma_{te}^{t}(G)$. Suppose that every vertex in D adjacent with one of D at least, then the number of edges of $G[D]$ equal $m_2 = \left\lceil \frac{|D|}{2} \right\rceil = \left\lceil \frac{\gamma_{te}^{t}(G)}{2} \right\rceil$.
Therefore, in general $m = m_1 + m_2$.
Case 2: To prove the upper bound let the two sub graphs $G[D]$and $G[V-D]$ are complete graphs. Let m_1 be the number of edges of $G[D]$ and m_2 be the number of edges of $G[V-D]$.
Thus, $m_1 = \frac{|D||D-1|}{2} = \frac{\gamma_{te}(\gamma_{te}-1)}{2}$ and $m_2 = \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{te})(n-\gamma_{te}-1)}{2}$ where $m_3 = 3|D| = 3\gamma_{te}$ is the number of edges between D and $V - D$.
Then, $m \leq m_1 + m_2 + m_3 = 3\gamma_{te} + \frac{\gamma_{te}^2 - \gamma_{te}}{2} + \frac{n^2 - n\gamma_{te} - n - n\gamma_{te} + \gamma_{te}^2 + \gamma_{te}}{2} = \binom{n}{2} + \gamma_{te}^2 - n\gamma_{te} + 3\gamma_{te}$.

Proposition 2.14: Let W_n be the wheel graph of order $(n \geq 3)$, then: W_n has no total triple effect domination.
Proof: Since every v in C_n has degree three if v adjacent with u in D, $(v, u \in D)$. Then, v dominates only two vertices, then W_n has no total triple effect domination.

Proposition 2.15: Let K_n be the complete graph $(n \geq 5)$, then K_n has total triple effect domination where $\gamma_{te}^{t}(K_n) = n - 3$.
Proof: Since every vertex in K_n is adjacent with all other vertices, so that every vertex in triple effect dominating set D dominates three vertices, then D must be having all vertices of K_n unless three vertices.

Proposition 2.16 : Let K_n be the complete graph, then K_n has inverse total triple effect domination if and only if $n = 5{,}6$, such that: $\gamma_{te}^{-t}(K_n) = n - 3$.
Proof: It is clear when $n = 5, 6$, then K_n has an inverse triple effect domination number equals to $n - 3$, by similar technique of [Proposition 2.15]. But if $n \geq 7$, then K_n has no inverse triple effect domination according to [For any graph G having order n and triple effect dominating set, if $\gamma_{te}(G) > \frac{n}{2}$, then G has no inverse triple effect domination].

Proposition 2.17: Let $K_{n,m}$ be the complete bipartite graph, then :
1. $K_{n,3}$ has no total triple effect domination if $n \geq 1$.
2. $K_{n,m}$ has total triple effect domination for all $n, m \geq 4$, such that:
$\gamma_{te}^{t}\left(K_{n,m}\right) = \gamma_{te}(K_{n,m}) = n + m - 6$
3. $K_{n,m}$ has an inverse total triple effect domination for all $4 \leq n, m \leq 6$, such that: $\gamma_{te}^{t}\left(K_{n,m}\right) = \gamma_{te}^{-t}\left(K_{n,m}\right) = \gamma_{te}^{-1}\left(K_{n,m}\right) = n + m - 6$.
Proof : Let $R_1$and R_2 are two disjoint subset of vertices of $K_{n,m}$ such that $|R_1| = n$ and $|R_2| = m$.
1. If $n = 1$, then $K_{1,3}$ has triple effect domination D contain isolated vertex inR_1, but that is contradiction with definition of total triple effect domination. Also if $n \geq 2$, therefore $K_{n,3}$ has no total triple effect domination in this case.
2. If $n, m > 3$, then D must be contains $n - 3$ vertices of V_1 dominate the three vertices of V_2. Also, D must be contains $m - 3$ vertices of V_2 dominate the three vertices of V_1 that belong to $V - D$. Hence, $\gamma_{te}^{t}\left(K_{n,m}\right) = n + m - 6$.
3. If $n, m > 3$, then D^{-1} must be contains $n - 3$ vertices of V_1 and $m - 3$ vertices of V_2 where all the $n - 3$ vertices will dominate the three vertices of V_2. Also, all $m - 3$ vertices of V_2 will dominate the three vertices of V_1. Hence, $\gamma_{te}^{-t}\left(K_{n,m}\right) = n + m - 6$.

3. Independent triple effect domination

The independent triple effect domination and the inverse independent triple effect domination. Some bounds and properties are putted and discussed on some known graphs.

Definition 3.1: Let G be a graph, a set D of $V(G)$ is an independent triple effect dominating set if D is a triple effect dominating set and $G[D]$ has no edges.

Definition 3.2: An independent triple effect dominating set is minimal if it has no proper independent triple effect dominating subset. D is minimum if its the cardinality is smallest overall independent triple effect dominating sets in G.

Definition 3.3: The independent triple effect domination number denoted by $\gamma_{te}^{i}(G)$ is the cardinality of the minimum independent triple effect dominating set in G. Such set is referred as γ_{te}^{i}-set.

Definition 3.4: Let G be a graph with γ_{te}^{i}-set D, a subset $D^{-1} \subseteq V - D$ is an inverse independent triple effect dominating set with respect to D, if $D^{-1}$is an independent triple effect dominating set.

Definition 3.5: An inverse independent triple effect dominating set D^{-1} is minimal if it has no proper inverse independent triple effect dominating subset. D^{-1} is minimum if its cardinality is smallest overall inverse independent triple effect dominating sets in G.

Definition 3.6: The inverse independent triple effect domination number denoted by $\gamma_{te}^{-i}(G)$ is the cardinality of the minimum inverse independent triple effect dominating set in G. Such set is referred as γ_{te}^{-i}-set. For example see figure 2.

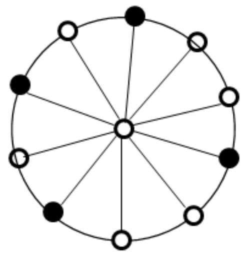

Figure 2: Independent triple effect dominating set of W_{10}.

Observation 3.7: Let G be a graph has γ_{te}^{-i}-set D, then :
1. $\deg(v) = 3$ for every $v \in D$.
2. If $\deg(v) \neq 3$, then $v \notin D \ \wedge v \notin D^{-1}$.

Theorem 3.8: Let $G(n, m)$ be a graph with independent triple effect domination, then:

$$3\gamma_{te}^{i}(G) \leq m \leq \binom{n}{2} + \frac{1}{2}\left(\gamma_{te}^{i}(G)\right)^2 + \frac{1}{2}(5 - 2n)\gamma_{te}^{i}(G)$$

Proof: Let D be the γ_{te}^{i}-set in G, then:

Case 1. From definition of triple effect dominating set there are exactly three edges incident between every vertex in D to $V - D$, then the number of edges between D and $V - D$ is $3|D| = 3\gamma_{te}^{i}(G)$. To prove lower bound, let $G[D]$ and $G[V - D]$ be two null graphs to be G contains as a few edges as possible. Then, $3\gamma_{te}^{i} \leq m$.

Case 2. Suppose that $G[V - D]$ be a complete subgraph having a maximum number of edges. Let m_1 be the number of edges of $G[V - D]$, then:

$$m_1 = \frac{|V-D||V-D-1|}{2} = \frac{\left(n-\gamma_{te}^{i}(G)\right)\left(n-\gamma_{te}^{i}(G)-1\right)}{2} = \frac{n^2-n-(2n+1)\gamma_{te}^{i}(G)+(\gamma_{te}^{i}(G))^2}{2}$$

Between every vertex of D to $V-D$, there exist exactly three vertices. Then, there are $m_2 = 3|D| = 3\gamma_{te}^{i}(G)$ and since $G[D]$ null graph, then $m_3 = zero$. Thus, the sum of edges in G is:

$m = m_1 + m_2 + m_3 \le \binom{n}{2} + \frac{1}{2}\left(\gamma_{te}^{i}(G)\right)^2 + \frac{1}{2}(5-2n)\gamma_{te}^{i}(G)$.

Proposition 3.9: Let $G(n,m)$ be a graph with an independent triple effect domination. Then, every independent triple effect dominating set D, is a minimal independent triple effect dominating set.

Proof : Any independent triple effect dominating set Din a graph G. Assume D is unminimal independent triple effect dominating set there is one vertex $v \in D$ or more such that $D - \{v\}$ is an independent triple effect dominating set. There are many cases to discuss :

Case 1. Suppose that at least vertex of the three vertices are dominated by vertex v is not dominated by any vertex other than v,then $D-v$ not independent triple effect dominating set and this is conflict.

Case 2. Let there is one or more vertices in $D-\{v\}$ dominate the three vertices in $V-D$ that are dominated by v. Then, we discuss which vertices are dominating vertex v. If $D-\{v\}$ has no vertex dominates v, therefor $D-\{v\}$ not an independent triple effect dominating set, this is a conflict. Else, $D-\{v\}$ dominates v by one vertex such as w or more. Thus, the four vertices dominated by vertex in $V-(D-\{v\})$. Thus, $D-\{v\}$ not an independent triple effect dominating set and that is contradiction. Then $D-\{v\}$ is not independent triple effect dominating set in all above cases. Hence, D is minimal independent triple effect dominating set.

Proposition 3.10: A complete graph K_n has an independent triple effect domination if and only if $n=4$, such that $\gamma_{te}^{i}(K_4) = \gamma_{te}^{-i}(K_4) = \gamma_{te}(K_4) = 1$.

Proof: Since $\gamma_{te}(K_n) = n-3$ by [5, Proposition 3.2], then $\gamma_{te}(K_4) = 1$, and this satisfy the definition of an independent triple effect domination. Therefore, $\gamma_{te}^{i}(K_4) = 1$, also $\gamma_{te}^{-i}(K_4) = 1$. If $n > 4$, every vertex in D is adjacent in D, therefore K_n has no independent domination.

Proposition 3.11 : For W_n wheel graph we have:

1. W_n has an independent triple effect domination such that: $\gamma_{te}^{i}(W_n) = \gamma_{te}(W_n) = \left\lceil \frac{n}{3} \right\rceil$.
2. W_n has an inverse independent triple effect domination such that:

$\gamma_{te}^{-i}(W_n) = \gamma_{te}^{-1}(W_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof :1. Since wheel graph $W_n = C_n + K_1$, let us table the vertices of W_n as: $v_1, v_2, \dots, v_{n+1}$ where $deg(v_i) = 3 \ \forall \ i = 1,2,\dots,n$ and $deg(v_{n+1}) = n$. Then:

Case i. If $n \equiv 0, 2 \ (mod\ 3)$, let D contains one vertex from every three vertices of C_n. Hence, $D = \{v_{3i-2}, i = 1,2,\dots,\left\lceil \frac{n}{3} \right\rceil\}$ is dominating set. All vertex in D dominates three vertices, v_{n+1} and added two vertices adjacent with it, except when $n \equiv 2 \ (mod\ 3)$, there are two vertices v_1 and v_{n-1} of D dominate v_n, v_{n+1} and another vertex. Therefore, D is $\gamma_{te}^{i} - set$ and $\gamma_{te}^{i} = |D| = \left\lceil \frac{n}{3} \right\rceil$.

Case ii. If $n \equiv 1 (mod\ 3)$, then we take $D = \left\{v_{3i-2}, i = 1,2,\dots,\left\lceil \frac{n}{3} \right\rceil - 1\right\} \cup \{v_{n-1}\}$.

Hence, D is $\gamma_{te}^{i} - set$ and $\gamma_{te} = |D| = \left\lceil \frac{n}{3} \right\rceil$, let $D^{'}$ is independent triple effect dominating set in G,such that $D^{'} \subseteq D$ and $\left|D^{'}\right| < |D|$,then there is at most vertex in $V-D$ don't dominated by any vertex of $D^{'}$.

Hence, D' is not independent triple effect dominating set and D is minimum independent triple effect dominating set.

2. let us table the vertices of W_n as: $v_1, v_2, \dots, v_{n+1}$.To choose a set D^{-1}, with respect to D that chosen in [10, Theorem 3.4]. The following two cases are obtained according to :

Case 1. If $n \equiv 0, 2\ (mod\ 3)$, let D^{-1} contains one vertex from every three consecutive vertices of C_n, where $D^{-1} = \{v_{3i-1}, i = 1,2, \dots, \left\lceil \frac{n}{3} \right\rceil\}$ is dominating set. Every vertex in D dominates three vertices, v_{n+1} and another two vertices adjacent with it, except when $n \equiv 2\ (mod\ 3)$, there are two vertices v_2 and v_n of D^{-1} dominate v_1, v_{n+1} and another vertex. Therefore, D^{-1} is $\gamma_{te}^{-i} -$ set and $\gamma_{te}^{-i} = |D^{-1}| = \left\lceil \frac{n}{3} \right\rceil$.

Case 2. If $n \equiv 1(mod\ 3)$, then we can take $D^{-1} = \left\{v_{3i-1}, i = 1,2, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\} \cup \{v_n\}$.Hence, D^{-1} is a $\gamma_{te}^{-i} - set$ and $\gamma_{te}^{-i} = |D^{-1}| = \left\lceil \frac{n}{3} \right\rceil$.Now, to prove that D^{-1} is minimum, let M is an inverse independent triple effect dominating set in G,such that $|M| < |D^{-1}|$,then there exist at least one vertex in $V - D$ don't dominated by any vertex of M. Hence, M is not inverse independent triple effect dominating set and D^{-1} is minimum inverse independent triple effect dominating set.

Proposition 3.12: For $K_{n,m}$ graph we have:

1. $\gamma_{te}^{i}(K_{n,m}) = \gamma_{te}(K_{n,m}) = n\ if\ and\ only\ if\ m = 3 \wedge n \geq 1.$
2. $\gamma_{te}^{-i}(K_{n,m}) = \gamma_{te}^{-1}(K_{n,m}) = 3\ if\ and\ only\ if\ n \wedge m = 3.$

Proof : Let $R_1$and R_2 be the two sets of vertices of $K_{n,m}$, such that $|R_1| = n$ and $|R_2| = m.$

1.Since $\gamma_{te}(K_{n,3}) = n$, for $n \geq 1$, and definition of an independent triple effect domination, we get $\gamma_{te}^{i}(K_{n,m}) = n.$

2. $\gamma_{te}^{-i}(K_{3,3}) = \gamma_{te}^{-1}(K_{3,3}) = 3$. It is clear $D^{-1} = V_1$ or $D^{-1} = V_2$, by definition of an independent triple effect domination.

Proposition 3.13: Let G be a graph :

1. $T_{m,n}$ a tadpole graph, then $\gamma_{te}^{i}(T_{n,m}) = 1$, if $m = 3, n = 1$.
2. $L_{m,n}$ a lollipop graph, then $\gamma_{te}^{i}(L_{n,m}) = 1$, if $m = 3, n = 1$.

Proof:

1. By the definition of tadpole graph there is a cycle C_m and path, then $T_{3,1}$ has C_3 and P_1. Hence, $D = \{v_2\}$ is minimum triple effect dominating set if $m \geq 4$ or $n \geq 2$, then $T_{m,n}$ has no γ_{te} −set according to Observation ($\deg(v) \geq 3\ \forall\ v \in D$).

2. If $m = 3$ and $n = 1$, then the result is given in (1). If $m \geq 4$ and $n \geq 2$, then any dominating set D of $L_{m,n}$ has one or more vertices. Hence $L_{m,n}$ has no triple effect dominating set.

4. Co-independent triple effect domination

The co-independent triple effect domination and the inverse co-independent triple effect domination. Some bounds and properties are putted and discussed on some known graphs.

Definition 4.1: Let G be a graph, a subset D of $V(G)$ is co-independent triple effect dominating set if D is a triple effect dominating set and $G[V - D]$ has no edges.

Definition 4.2: A co-independent triple effect dominating set is minimal if it has no proper co-independent triple effect dominating subset. D is minimum if its the cardinality is smallest overall co-independent triple effect dominating sets in G.

Definition 4.3: The co-independent triple effect domination number denoted by $\gamma_{te}^{coi}(G)$ is the cardinality of the minimum co-independent triple effect dominating set in G. Such set is referred as γ_{te}^{coi}-set.

Definition 4.4: Let G be a graph with γ_{te}^{coi}-set D, a subset $D^{-1} \subseteq V - D$ is an inverse co-independent triple effect dominating set with respect to D, if $D^{-1}$is co-independent triple effect dominating set.

Definition 4.5: An inverse co-independent triple effect dominating set D^{-1} is minimal if it has no proper inverse co-independent triple effect dominating subset. D^{-1} is minimum if its cardinality is smallest overall inverse co-independent triple effect dominating sets in G.

Definition 4.6: The inverse co-independent triple effect domination number denoted by $\gamma_{te}^{-coi}(G)$ is the cardinality of the minimum inverse co-independent triple effect dominating set in G. Such set is referred as γ_{te}^{-coi}-set. For example see figure 3.

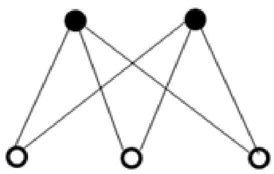

Figure 3: Co-independent triple effect dominating set of $K_{2,3}$.

Observation 4.7: Let D be a γ_{te}^{coi} $-$set, and D^{-1} be γ_{te}^{-coi} $-$set in G, then $\deg(v) = 3$ for every $v \in D \text{ or } D^{-1}$.

Proposition 4.8: Let $G(n,m)$ be a graph with γ_{te}^{coi} $-$set D, if G has γ_{te}^{-coi} $-$set D^{-1}, then $D^{-1} = V - D$ and $\gamma_{te}^{coi}(G) + \gamma_{te}^{-coi}(G) = n$.

Proof: Since D is a γ_{te}^{coi} $-$set of G, then $G[V - D]$ is null graph. If there is vertex $v \in V - D$ such that $v \notin D^{-1}$, then v is not dominated by any vertex of D^{-1}, since it has no neighbor in $V - D$ where $D^{-1} \subseteq V - D$. Thus, $D^{-1} = V - D$ and $|D^{-1}| = |V - D| = n - \gamma_{te}^{coi}(G)$ by [6, condition 2 of Theorem (2.14)].

Theorem 4.9: Let $G(n,m)$ be a graph with co-independent triple effect domination, then:

$$3\gamma_{te}^{coi}(G) \leq m \leq \frac{1}{2}\left(\gamma_{te}^{coi}(G)\right)^2 + \frac{5}{2}(5 - 2n)\gamma_{te}^{coi}(G)$$

Proof: Let D be the γ_{te}^{coi} $-$set in G, then:

Case 1. Let $G[D]$ be a null graph, where $G[V - D]$ has no edges. Now there are three edges from every $u \in D$ and $V - D$. Thus, the number of edges between D and $V - D$ is $|D| = 3\gamma_{te}^{i}(G)$. Hence, $3\gamma_{te}^{coi} \leq m$.

Case 2. Let $G[D]$ be a complete subgraph having a maximum number of edges. Let m_1 be the number of edges of $G[D]$, then:

$m_1 = \frac{|D||D-1|}{2} = \frac{(\gamma_{te}^{coi}(G))^2 - \gamma_{te}^{coi}(G)}{2}$. Since $G[V-D]$ is null, then $m_3 = zero$. Thus, the number of an edges in G is $m = m_1 + m_2 \leq \frac{1}{2}\left(\gamma_{te}^{coi}(G)\right)^2 - \frac{1}{2}\gamma_{te}^{coi}(G) + 3\gamma_{te}^{coi}(G)$.

Proposition 4.10: For any graph G, then:

1. $K_n, (n \geq 4)$ has no co-independent triple effect domination.

2. $W_n, (n \geq 3)$ has no co-independent triple effect domination.

Proof: 1. Since every vertices in $V-D$ are adjacent, therefore K_n has no co-independent triple effect domination.

2. Since every vertex in $V-D$ has neighborhood in $V-D$, therefore W_n has no co-independent triple effect domination.

Proposition 4.11: For $K_{n,m}$ graph we have:

1. $\gamma_{te}^{coi}(K_{n,m}) = \gamma_{te}(K_{n,m}) = n \text{ if and only if } m = 3 \wedge n \geq 1$.

2. $\gamma_{te}^{-coi}(K_{n,m}) = \gamma_{te}^{-1}(K_{n,m}) = 3 \text{ if and only if } n \wedge m = 3$.

Proof : Let $R_1$and R_2 be the two sets of vertices of $K_{n,m}$, such that $|R_1| = n$ and $|R_2| = m$.

1. See proof Proposition (3.12 case 1).

2. See proof Proposition (3.12 case 2).

5. Connected triple effect domination

By adding a new condition on a subgraph $G[D]$ get a new type of domination said the connected domination and find its inverse and studied it and putted bounds and properties and applied on some graphs.

Definition 5.1: A subset $D \subseteq V(G)$ is connected triple effect dominating set of G, if D is triple effect dominating set and $G[D]$ is a connected induced subgraph.

Definition 5.2: A connected triple effect dominating set is minimal if it has no proper connected triple effect dominating subset. D is minimum if its the cardinality is smallest overall connected triple effect dominating sets in G.

Definition 5.3: The connected triple effect domination number denoted by $\gamma_{te}^{c}(G)$ is the cardinality of the minimum connected triple effect dominating set in G. Such set is referred as γ_{te}^{c}-set.

Definition 5.4: Let G be a graph with γ_{te}^{c}-set D, a subset $D^{-1} \subseteq V-D$ is an inverse connected triple effect dominating set with respect to D, if $D^{-1}$is connected triple effect dominating set.

Definition 5.5: An inverse connected triple effect dominating set D^{-1} is minimal if it has no proper inverse connected triple effect dominating subset. D^{-1} is minimum if its cardinality is smallest overall inverse connected triple effect dominating sets in G.

Definition 5.6: The inverse connected triple effect domination number denoted by $\gamma_{te}^{-c}(G)$ is the cardinality of the minimum inverse connected triple effect dominating set in G. Such set is referred as γ_{te}^{-c}-set. For example see figure 4.

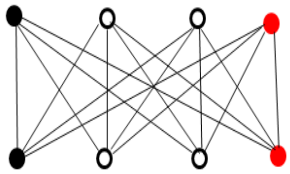

Figure 4. Connected triple effect dominating set and its inverse in $K_{4,4}$.

Theorem 5.7 : Let $G(n, m)$ be a graph with connected triple effect domination, then:
$4\gamma_{te}^{c}(G) - 1 \leq m \leq \binom{n}{2} + \left(\gamma_{te}^{c}(G)\right)^2 - n\,\gamma_{te}^{c}(G) + 3\gamma_{te}^{c}(G)$
Proof: Let D be the γ_{te}^{c} −set in G, then:
Case 1. To prove the lower bound, the number of edge between G$[D]$ and G$[\ V - D]$ is $m_1 = 3|D| = 3\gamma_{te}^{c}(G)$. Since D is a triple effect dominating set, a subgraph $G[D]$ should contain as few edges as posiple to become a connected, it is $m_2 = \gamma_{te}^{c}(G) - 1$, and $m_3 = m_3(G[V - D]) = zero$, Thus, $m = m_1 + m_2 + m_3 \geq 4\gamma_{te}^{c}(G) - 1$.
Case 2. Similar to proof of Proposition (2.13) case 2.

Proposition 5.8: Let G be a graph, then:
1. $\gamma_{te}^{c}(K_n) = \gamma_{te}^{-c}(K_n) = n - 3, n \geq 4$.
2. $\gamma_{te}^{c}\left(K_{n,m}\right) = \gamma_{te}^{-c}\left(K_{n,m}\right) = n + m - 6, n \wedge m > 3$.
3. W_n has no connected triple effect domination.
Proof: 1. Similar to proof of Proposition (2.15) and Proposition (2.16).
2. Similar to proof of Proposition (2.17 case 2).
3. Since $G[D]$ disconnected sub graph of G.

Observation 5.9: Since W_n has no connected triple effect domination, then W_n has no an inverse connected triple effect domination.

6. Doubly connected triple effect domination
The doubly connected domination and the inverse doubly connected domination are introduced by adding a condition on both induced subgraphs $G[D]$ and $G[V - D]$. Some bounds and properties are studied an applied on some graphs.

Definition 6.1 : Let G be a graph, a subset $D \subseteq V(G)$ is doubly connected triple effect dominating set of G, if D is triple effect dominating set and both $G[D]$ and $G[V - D]$ are a connected subgraphs of G.

Definition 6.2: A doubly connected triple effect dominating set is minimal if it has no proper doubly connected triple effect dominating subset. D is minimum if its the cardinality is smallest overall doubly connected triple effect dominating sets in G.
Definition 6.3: The doubly connected triple effect domination number denoted by $\gamma_{te}^{cc}(G)$ is the cardinality of the minimum doubly connected triple effect dominating set in G. Such set is referred as γ_{te}^{cc}-set.

Definition 6.4: Let G be a graph with γ_{te}^{cc}-set D, a subset $D^{-1} \subseteq V - D$ is an inverse doubly connected triple effect dominating set with respect to D, if $D^{-1}$is a doubly connected triple effect dominating set.

Definition 6.5: An inverse doubly connected triple effect dominating set D^{-1} is minimal if it has no proper inverse doubly connected triple effect dominating subset. D^{-1} is minimum if its cardinality is smallest overall inverse doubly connected triple effect dominating sets in G.

Definition 6.6: The inverse doubly connected triple effect domination number denoted by $\gamma_{te}^{-cc}(G)$ is the cardinality of the minimum inverse doubly connected triple effect dominating set in G. Such set is referred as γ_{te}^{-cc}-set. For example see figure 5.

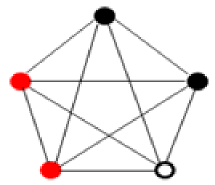

Figure.5. Doubly connected triple effect dominating set and its invers in K_5.

Theorem 6.7: Let (n, m) $(n \geq 4)$, be a graph with a doubly connected triple effect domination number $\gamma_{te}^{cc}(G)$, then $1 \leq \gamma_{te}^{cc}(G) \leq n - 3$.
Proof : Suppose that $n \geq 4$, then D must be contains at least one vertex, since if D has one vertex then this vertex dominates three vertices.

Theorem 6.8: Let $G(n, m)$ be a graph has a doubly connected triple effect domination, then: $3\gamma_{te}^{cc}(G) + n - 2 \leq m \leq \binom{n}{2} + \left(\gamma_{te}^{cc}(G)\right)^2 - n\,\gamma_{te}^{cc}(G) + 3\gamma_{te}^{cc}(G)$
Proof: Let D be the γ_{te}^{cc} −set in G, then:
Case 1. To prove the lower bound, since $G[D]$ and $G[V - D]$ be a connected graphs. Then, let $m_1 = m(G[D]) = |D| - 1 = \gamma_{te}^{cc} - 1$. And $m_2 = m(G[V - D]) = |V - D| - 1 = n - \gamma_{te}^{cc} - 1$ to be G has a few edges as possible. Now by the definition of doubly connected triple effect domination, there are exactly three edges between every vertex in D to $V - D$. Then, $m_3 = |D| = 3\gamma_{te}^{cc}$. Therefore, in general $m = m_1 + m_2 + m_2 \geq 3\gamma_{te}^{cc} + n - 2$
Case 2. By similar way of [Proposition 2.13 case 2].

Proposition 6.9: Let G be a graph, then:
1. $\gamma_{te}^{cc}(K_n) = \gamma_{te}^{-cc}(K_n) = n - 3, n \geq 4.$
2. $\gamma_{te}^{cc}\left(K_{n,m}\right) = \gamma_{te}^{-cc}\left(K_{n,m}\right) = n + m - 6, (n \wedge m > 3).$
Proof: 1. Similar to proof of Proposition (2.15) and Proposition (2.16).
2. Similar to proof of Proposition (2.17 case 2).

4 Conclusion

In this paper, five types of the triple effect domination are introduced with their invers. Several bounds and properties are proved. An applications of these types of domination are given on some known graphs.

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