



New application of the Cauchy operator on the homogeneous Rogers–Szegő polynomials

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Abstract

This paper is mainly concerned with using the Cauchy operator $T(a, b; D_q)$ in proving identities that involve the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$. We introduce some operator identities for the Cauchy operator and represent the homogeneous Rogers–Szegő polynomials by the Cauchy operator. Also we use the Cauchy operator to derive the basic identities for $h_n(x, y|q)$: generating function, Mehler’s formula and Rogers formula. Then, we give several extended identities for $h_n(x, y|q)$ such as extended generating function, extended Mehler’s formula, extended Rogers formula and other extended identities.

Keywords Rogers–Szegő polynomials · homogenous Rogers–Szegő polynomials · Cauchy operator · Generating function · Mehler’s formula, Rogers formula · q -Difference equation

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1 Introduction and notation

Let us review some common notation and terminology for basic hypergeometric series in [14], where $|q| < 1$. The q -shifted factorial is given by

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$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (1.1)$$

Notice that

$$(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty.$$

$$(a; q)_{n+k} = (a; q)_k (aq^k; q)_n.$$

We shall use the following notations of multiple q -shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The q -binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The generalized basic hypergeometric series is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \cdots (b_s; q)_n} \times \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n,$$

where $q \neq 0$ when $r > s + 1$. Notice that

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_{r+1}; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \cdots (b_r; q)_n} x^n.$$

The Cauchy identity is defined as

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \quad (1.2)$$

Euler found the following special case of the Cauchy identity (1.2):

$$\sum_{n=0}^{\infty} \frac{x^k}{(q; q)_k} = \frac{1}{(x; q)_\infty}, \quad |x| < 1. \quad (1.3)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q; q)_k} = (x; q)_\infty. \quad (1.4)$$

The Cauchy polynomial is defined by

$$P_n(x, y) = (y/x; q)_n x^n = (x - y)(x - qy) \cdots (x - q^{n-1}y).$$

The Cauchy polynomial is the homogeneous form of the q -shifted factorial (1.1). It has the following generating function (the homogeneous form of the Cauchy identity):

$$\sum_{k=0}^{\infty} P_k(x, y) \frac{t^k}{(q; q)_k} = \frac{(yt; q)_{\infty}}{(xt; q)_{\infty}}, \quad |xt| < 1.$$

Setting $y = 0$ and $x = 0$, the Cauchy identity becomes Euler’s identities (1.3) and (1.4), respectively.

The classical Rogers–Szegő polynomial is defined by

$$h_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{1.5}$$

which has following generating function:

$$\sum_{n=0}^{\infty} h_n(x|q) \frac{t^n}{(q; q)_n} = \frac{1}{(t, xt; q)_{\infty}}, \quad \max \{|t|, |xt|\} < 1. \tag{1.6}$$

Mehler’s formula for $h_n(x|q)$ is

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(xyt^2; q)_{\infty}}{(t, xt, yt, xyt; q)_{\infty}}, \tag{1.7}$$

where $\max \{|t|, |xt|, |yt|, |xyt|\} < 1$ and the Rogers formula of $h_n(x|q)$ is

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(xst; q)_{\infty}}{(t, s, xt, xs; q)_{\infty}}, \tag{1.8}$$

where $\max \{|s|, |t|, |xs|, |xt|\} < 1$. We will call the identities (1.6), (1.7) and (1.8) as the basic identities for $h_n(x|q)$. The Rogers–Szegő polynomials play an important role in the theory of the orthogonal polynomials, particularly in the study of the Askey–Wilson polynomials [6–8,17,18].

Hahn polynomials are defined as follows [5,15,16,23]:

$$\phi_n^{(a)}(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k. \tag{1.9}$$

In 1965, Al-Salam and Carlitz [5] obtained the following results:

Theorem 1.1 [5]

The generating function for $\phi_n^{(a)}(x)$ is

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x) \frac{t^n}{(q; q)_n} = \frac{(axt; q)_{\infty}}{(t, xt; q)_{\infty}}, \quad \max\{|t|, |xt|\} < 1. \tag{1.10}$$

Mehler’s formula for $\phi_n^{(a)}(x)$ is

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)\phi_n^{(b)}(y) \frac{t^n}{(q; q)_n} = \frac{(axt, byt; q)_{\infty}}{(t, xt, yt; q)_{\infty}} {}_3\phi_2\left(\begin{matrix} a, b, t \\ axt, byt \end{matrix}; q, xyt\right), \tag{1.11}$$

where $\max\{|t|, |xt|, |yt|, |xyt|\} < 1$.

The q -differential operator D_q is defined by

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.12}$$

The following identities are easy to verify [9]:

$$D_q^k \left\{ \frac{1}{(at; q)_{\infty}} \right\} = \frac{t^k}{(at; q)_{\infty}}, \tag{1.13}$$

$$D_q^n \left\{ \frac{(av; q)_{\infty}}{(at; q)_{\infty}} \right\} = t^n (v/t; q)_n \frac{(avq^n; q)_{\infty}}{(at; q)_{\infty}}. \tag{1.14}$$

In 1997, Chen and Liu [9] defined the q -exponential operator $T(bD_q)$ as follows:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}.$$

They used the q -exponential operator $T(D_q)$ to represent the classical Rogers–Szegő polynomials $h_n(x|q)$ to derive Mehler’s formula and Rogers formula for $h_n(x|q)$.

In 2003, Chen et al. [11] introduced the homogeneous q -difference operator D_{xy} on functions in two variables as follows:

$$D_{xy}\{f(x, y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y},$$

which turns out to be suitable for dealing with the homogeneous form of the q -binomial theorem. They derived the Leibniz formula for this operator. Based on the homogeneous q -difference operator, they built up the homogeneous q -shift operator $\mathbb{E}(D_{xy})$ as follows:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q; q)_k}.$$

Also they introduced the homogeneous Rogers–Szegő polynomials as

$$h_n(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y), \tag{1.15}$$

with the generating function

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}}, \quad \max\{|t|, |xt|\} < 1. \tag{1.16}$$

The Rogers–Szegő polynomials $h_n(x|q)$ are special case of $h_n(x, y|q)$ when y is set to zero.

In 2007, Chen et. al. [12] used the q -exponential operator $T(bD_q)$ and the homogeneous q -shift operator $\mathbb{E}(D_{xy})$ to recover Mehler’s formula and derive the Rogers formula for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$.

Theorem 1.2 [12] *Mehler’s formula for $h_n(x, y|q)$ is as follows:*

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q) \frac{t^n}{(q; q)_n} &= \frac{(yt, xvt; q)_{\infty}}{(t, xt, xut; q)_{\infty}} {}_3\phi_2 \\ &\times \left(\begin{matrix} y, xt, v/u \\ yt, xvt \end{matrix}; q, ut \right), \end{aligned} \tag{1.17}$$

where $\max\{|t|, |xt|, |ut|, |xut|\} < 1$.

The Roger’s formula for $h_n(x, y|q)$ is as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} &= \frac{(ys; q)_{\infty}}{(s, xs, xt; q)_{\infty}} {}_2\phi_1 \\ &\times \left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t \right), \end{aligned} \tag{1.18}$$

where $\max\{|t|, |s|, |xt|, |xs|\} < 1$.

In 2008, Chen and Gu [10] introduced the Cauchy operator for basic hypergeometric series as follows:

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n. \tag{1.19}$$

The Cauchy operator is the reminiscent of the Cauchy identity (1.2). Compared with the q -exponential operator $T(bD_q)$, the Cauchy operator (1.19) involves two parameters. Clearly, the q -exponential operator $T(bD_q)$ can be considered as a special case of the Cauchy operator (1.19) for $a = 0$. They obtained the following operator identities, where its assumed that the Cauchy operator $T(a, b; D_q)$ acts on the parameter c :

Theorem 1.3 [10] Let D_q and $T(a, b; D_q)$ be defined as in (1.12) and (1.19), respectively. Then

$$T(a, b; D_q)\{c^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k b^k c^{n-k}. \tag{1.20}$$

$$T(a, b; D_q) \left\{ \frac{1}{(ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}, \quad |bt| < 1. \tag{1.21}$$

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, ct \\ abt \end{matrix}; q, bs \right),$$

$$\max\{|bs|, |bt|\} < 1. \tag{1.22}$$

$$T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a, cs, v/t \\ abs, cv \end{matrix}; q, bt \right),$$

$$\max\{|bs|, |bt|\} < 1. \tag{1.23}$$

They found that the Cauchy operator is suitable for studying the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ because of the following fact:

$$\lim_{c \rightarrow 1} T(y/x, x; D_q)\{c^n\} = h_n(x, y|q). \tag{1.24}$$

They used (1.24) to derive Mehler’s formula and Rogers formula for $h_n(x, y|q)$.

In 2010, Saad and Sukhi [20] gave another formula for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ as

$$h_n(x, y|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (y; q)_k x^{n-k}. \tag{1.25}$$

In 2014, Zhou and Luo [23] obtained some new generating functions for q -Hahn polynomials and gave their proofs based on the homogeneous q -difference operator $\mathbb{E}(D_{xy})$. They obtained the Rogers formula for $\phi_n^{(a)}(x)$ as follows:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{m+n}^{(a)}(x) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(axs; q)_\infty}{(s, xs, xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} xa, xs \\ axs \end{matrix}; q, t \right), \tag{1.26}$$

where $\max\{|s|, |t|, |xs|, |xt|\} < 1$.

Notice that

$$\phi_n^{(y/x)}(x) = h_n(x, y|q). \tag{1.27}$$

$$\phi_n^{(y)}(1/x) = x^{-n} h_n(x, y|q). \tag{1.28}$$

The homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ introduced by Chen et al. [11] are equivalent to the Hahn polynomials $\phi_n^{(a)}(x)$ which were first studied by

Hahn [15] and then by Al-Salam and Carlitz [5] and Zhou and Luo [23]. The results of Al-Salam and Carlitz [5] and Zhou and Luo [23] for the polynomials $\phi_n^{(a)}(x)$ can be used to get results for the polynomials $h_n(x, y|q)$ and vice versa. For example, setting $a = y/x$ in the generating function for $\phi_n^{(a)}(x)$ (1.10), we get the generating function for $h_n(x, y|q)$ (1.16). Also, setting $a = y, x \rightarrow 1/x, t \rightarrow xt, b = v/u, y = u$ in the Mehler's formula for $\phi_n^{(a)}(x)$ (1.11), we get Mehler's formula for $h_n(x, y|q)$ (1.17). Specifying $y = ax$ in the Rogers formula for $h_n(x, y|q)$ (1.18), we get Rogers formula for $\phi_n^{(a)}(x)$ (1.26).

In this paper, we use the Cauchy operator to represent the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ in the form (1.25). We use our operator representation to give a short and simple operator proof for the basic identities for $h_n(x, y|q)$. Also we give some extensions for the basic identities. The technique of using operators in representing q -polynomials is more used as in [1–4, 12, 13, 19–22].

2 Operator identities for the Cauchy operator

In this section, we derive some operator identities for the Cauchy operator $T(a, b; D_q)$. These identities are very important for proving the extended identities for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$.

Theorem 2.1 For $n \in N$, we have

$$T(a, b; D_q) \left\{ \frac{c^n}{(cs, ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} \sum_{l=0}^\infty \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_{j+l} (cs; q)_j}{(abt; q)_{j+l} (q; q)_l} \times b^{j+l} c^{n-j} s^l, \tag{2.1}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned} & T(a, b; D_q) \left\{ \frac{c^n}{(cs, ct; q)_\infty} \right\} \\ &= \sum_{k=0}^\infty \frac{(a; q)_k b^k}{(q; q)_k} D_q^k \left\{ \frac{c^n}{(cs, ct; q)_\infty} \right\} \\ &= \sum_{k=0}^\infty \frac{(a; q)_k b^k}{(q; q)_k} \sum_{j=0}^k q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} D_q^j \{c^n\} D_q^{k-j} \left\{ \frac{1}{(csq^j, ctq^j; q)_\infty} \right\} \\ &= \sum_{k=0}^\infty \frac{(a; q)_k b^k}{(q; q)_k} \sum_{j=0}^k q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q; q)_n}{(q; q)_{n-j}} c^{n-j} D_q^{k-j} \left\{ \frac{1}{(csq^j, ctq^j; q)_\infty} \right\} \\ &= \sum_{j=0}^n \sum_{k=0}^\infty \frac{(a; q)_{k+j} b^{k+j}}{(q; q)_k} q^{-jk} \begin{bmatrix} n \\ j \end{bmatrix} c^{n-j} D_q^k \left\{ \frac{1}{(csq^j, ctq^j; q)_\infty} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \sum_{k=0}^{\infty} \frac{(aq^j; q)_k (bq^{-j})^k}{(q; q)_k} D_q^k \left\{ \frac{1}{(csq^j, ctq^j; q)_{\infty}} \right\} \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} T(aq^j, bq^{-j}; D_q) \left\{ \frac{1}{(csq^j, ctq^j; q)_{\infty}} \right\} \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \frac{(abtq^j; q)_{\infty}}{(bt, csq^j, ctq^j; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} aq^j, ctq^j \\ abtq^j \end{matrix}; q, bs \right) \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \frac{(abtq^j; q)_{\infty}}{(bt, csq^j, ctq^j; q)_{\infty}} \sum_{l=0}^{\infty} \frac{(aq^j, ctq^j; q)_l}{(abtq^j, q; q)_l} (bs)^l \\
 &= \frac{(abt; q)_{\infty}}{(bt, cs, ct; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_{j+l} (cs; q)_j}{(abt; q)_{j+l} (q; q)_l} b^{j+l} c^{n-j} s^l.
 \end{aligned}$$

□

Setting $n = 0$ in (2.1), we get (1.22).

Setting $s = 0$ in (2.1), we get the following corollary:

Corollary 2.1 For $n \in N$, we have

$$T(a, b; D_q) \left\{ \frac{c^n}{(ct; q)_{\infty}} \right\} = \frac{(abt; q)_{\infty}}{(bt, ct; q)_{\infty}} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_j}{(abt; q)_j} b^j c^{n-j}, \quad (2.2)$$

where $|bt| < 1$.

Setting $n = 0$ in (2.2), we get (1.21).

Theorem 2.2 For $n \in N$, we have

$$\begin{aligned}
 T(a, b; D_q) \left\{ \frac{c^n (cv; q)_{\infty}}{(cs, ct; q)_{\infty}} \right\} &= \frac{(abs, cv; q)_{\infty}}{(bs, cs, ct; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \\
 &\quad \times \frac{(a, cs; q)_{j+l} (ct; q)_j (v/t; q)_l}{(abs, cv; q)_{j+l} (q; q)_l} \\
 &\quad \times b^{j+l} c^{n-j} t^l, \quad (2.3)
 \end{aligned}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned}
 &T(a, b; D_q) \left\{ c^n \frac{(cv; q)_{\infty}}{(cs, ct; q)_{\infty}} \right\} \\
 &= \sum_{k=0}^{\infty} \frac{(a; q)_k b^k}{(q; q)_k} D_q^k \left\{ c^n \frac{(cv; q)_{\infty}}{(cs, ct; q)_{\infty}} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} \frac{(a; q)_k b^k}{(q; q)_k} \sum_{j=0}^k q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} D_q^j \{c^n\} D_q^{k-j} \left\{ \frac{(cvq^j; q)_{\infty}}{(csq^j, ctq^j; q)_{\infty}} \right\} \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \sum_{k=0}^{\infty} \frac{(aq^j; q)_k (bq^{-j})^k}{(q; q)_k} D_q^k \left\{ \frac{(cvq^j; q)_{\infty}}{(csq^j, ctq^j; q)_{\infty}} \right\} \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} T(aq^j, bq^{-j}; D_q) \left\{ \frac{(cvq^j; q)_{\infty}}{(csq^j, ctq^j; q)_{\infty}} \right\} \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \frac{(absq^j, cvq^j; q)_{\infty}}{(bs, csq^j, ctq^j; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} aq^j, csq^j, v/t \\ absq^j, cvq^j \end{matrix}; q, bt \right) \\
 &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (a; q)_j b^j c^{n-j} \frac{(absq^j, cvq^j; q)_{\infty}}{(bs, csq^j, ctq^j; q)_{\infty}} \sum_{l=0}^{\infty} \frac{(aq^j, csq^j, v/t; q)_l}{(absq^j, cvq^j, q; q)_l} (bt)^l \\
 &= \frac{(abs, cv; q)_{\infty}}{(bs, cs, ct; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, cs; q)_{j+l} (ct; q)_j (v/t; q)_l}{(abs, cv; q)_{j+l} (q; q)_l} b^{j+l} c^{n-j} t^l.
 \end{aligned}$$

□

Setting $v = 0$ in (2.3), we get (2.1). Setting $v = s = 0$ in (2.3), we get (2.2) and setting $n = 0$ in (2.3) we get (1.23).

Theorem 2.3 *We have*

$$\begin{aligned}
 T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cv; q)_{\infty}} \right\} &= \frac{(abt; q)_{\infty}}{(bt, cs, ct, cv; q)_{\infty}} \sum_{k,j=0}^{\infty} \frac{(a, ct; q)_{k+j} (cs; q)_k}{(abt; q)_{k+j}} \\
 &\quad \times \frac{(bs)^j (bv)^k}{(q; q)_j (q; q)_k}, \tag{2.4}
 \end{aligned}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned}
 &T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cv; q)_{\infty}} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cv; q)_{\infty}} \right\} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k; q)_{\infty}} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{v^k}{(cv; q)_{\infty}} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k; q)_{\infty}} \right\} \\
 &= \frac{1}{(cv; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a; q)_k (bv)^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(aq^k; q)_n (bq^{-k})^n}{(q; q)_n} D_q^n \left\{ \frac{1}{(csq^k, ctq^k; q)_{\infty}} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(cv; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (bv)^k}{(q; q)_k} T(aq^k, bq^{-k}; D_q) \left\{ \frac{1}{(csq^k, ctq^k; q)_\infty} \right\} \\
 &= \frac{1}{(cv; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (bv)^k}{(q; q)_k} \frac{(abtq^k; q)_\infty}{(bt, csq^k, ctq^k; q)_\infty} {}_2\phi_1 \left(\begin{matrix} aq^k, ctq^k \\ abtq^k \end{matrix}; q, bs \right) \\
 &= \frac{1}{(cv; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (bv)^k}{(q; q)_k} \frac{(abtq^k; q)_\infty}{(bt, csq^k, ctq^k; q)_\infty} \sum_{j=0}^\infty \frac{(aq^k, ctq^k; q)_j}{(abtq^k, q; q)_j} (bs)^j \\
 &= \frac{(abt; q)_\infty}{(bt, cs, ct, cv; q)_\infty} \sum_{k,j=0}^\infty \frac{(a, ct; q)_{k+j} (cs; q)_k}{(abt; q)_{k+j}} \frac{(bs)^j}{(q; q)_j} \frac{(bv)^k}{(q; q)_k}.
 \end{aligned}$$

□

Setting $v = 0$ in (2.4), we get (1.22).

Theorem 2.4 *We have*

$$\begin{aligned}
 T(a, b; D_q) &\left\{ \frac{(cv; q)_\infty}{(cs, ct, cu; q)_\infty} \right\} \\
 &= \frac{(abs, cv; q)_\infty}{(bs, cs, ct, cu; q)_\infty} \\
 &\quad \times \sum_{k,j=0}^\infty \frac{(a, cs; q)_{k+j} (v/t; q)_j (ct; q)_k}{(abs, cv; q)_{k+j}} \frac{(bt)^j}{(q; q)_j} \\
 &\quad \times \frac{(bu)^k}{(q; q)_k},
 \end{aligned} \tag{2.5}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned}
 &T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(cs, ct, cu; q)_\infty} \right\} \\
 &= \sum_{n=0}^\infty \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cu; q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\} \\
 &= \sum_{n=0}^\infty \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^k}{(cu; q)_\infty} D_q^{n-k} \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\} \\
 &= \frac{1}{(cu; q)_\infty} \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{(a; q)_{n+k} b^{n+k}}{(q; q)_k (q; q)_n} q^{-nk} u^k D_q^n \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\} \\
 &= \frac{1}{(cu; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (bu)^k}{(q; q)_k} \sum_{n=0}^\infty \frac{(aq^k; q)_n (bq^{-k})^n}{(q; q)_n} D_q^n \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(cu; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bu)^k}{(q; q)_k} T(aq^k, bq^{-k}; D_q) \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \right\} \\
 &= \frac{1}{(cu; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bu)^k}{(q; q)_k} \frac{(absq^k, cvq^k; q)_\infty}{(bs, csq^k, ctq^k; q)_\infty} {}_3\phi_2 \left(\begin{matrix} aq^k, csq^k, v/t \\ absq^k, cvq^k \end{matrix}; q, bt \right) \\
 &= \frac{1}{(bs, cu; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bu)^k}{(q; q)_k} \frac{(absq^k, cvq^k; q)_\infty}{(csq^k, ctq^k; q)_\infty} \\
 &\quad \times \sum_{j=0}^{\infty} \frac{(aq^k, csq^k, v/t; q)_j}{(absq^k, cvq^k, q; q)_j} (bt)^j \\
 &= \frac{(abs, cv; q)_\infty}{(bs, cs, ct, cu; q)_\infty} \sum_{k,j=0}^{\infty} \frac{(a, cs; q)_{k+j} (v/t; q)_j (ct; q)_k}{(abs, cv; q)_{k+j}} \frac{(bt)^j}{(q; q)_j} \frac{(bu)^k}{(q; q)_k}.
 \end{aligned}$$

□

Setting $v = 0$ and replace s with t in (2.5), we get (2.4). Setting $v = u = 0$ (2.5), we get identity (1.22).

Theorem 2.5 *We have*

$$\begin{aligned}
 &T(a, b; D_q) \left\{ \frac{(cv, cw; q)_\infty}{(cs, ct, cu, cz; q)_\infty} \right\} \\
 &= \frac{(abs, cv, cw; q)_\infty}{(bs, cs, ct, cu, cz; q)_\infty} \sum_{j,k,l=0}^{\infty} \frac{(a, cs; q)_{j+k+l} (ct; q)_{k+l} (cu, w/z; q)_k (v/t; q)_j}{(abs, cv; q)_{j+k+l} (cw; q)_k} \\
 &\quad \times \frac{(bt)^j}{(q; q)_j} \frac{(bz)^k}{(q; q)_k} \frac{(bu)^l}{(q; q)_l}, \tag{2.6}
 \end{aligned}$$

where $\max\{|bs|, |bt|\} < 1$.

Proof

$$\begin{aligned}
 &T(a, b; D_q) \left\{ \frac{(cv, cw; q)_\infty}{(cs, ct, cu, cz; q)_\infty} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \\
 &\quad \times \left\{ \frac{(cw; q)_\infty}{(cz; q)_\infty} \right\} D_q^{n-k} \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k, cuq^k; q)_\infty} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} z^k (w/z; q)_k \frac{(cwq^k; q)_\infty}{(cz; q)_\infty} D_q^{n-k} \\
 &\quad \times \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k, cuq^k; q)_\infty} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(cw; q)_\infty}{(cz; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (w/z; q)_k (bz)^k}{(cw; q)_k (q; q)_k} \sum_{n=0}^\infty \frac{(aq^k; q)_n (bq^{-k})^n}{(q; q)_n} \\
 &\quad \times D_q^n \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k, cuq^k; q)_\infty} \right\} \\
 &= \frac{(cw; q)_\infty}{(cz; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (w/z; q)_k (bz)^k}{(cw; q)_k (q; q)_k} T(aq^k, bq^{-k}; D_q) \\
 &\quad \times \left\{ \frac{(cvq^k; q)_\infty}{(csq^k, ctq^k, cuq^k; q)_\infty} \right\} \\
 &= \frac{(cw; q)_\infty}{(cz; q)_\infty} \sum_{k=0}^\infty \frac{(a; q)_k (w/z; q)_k (bz)^k}{(cw; q)_k (q; q)_k} \frac{(absq^k, cvq^k; q)_\infty}{(bs, csq^k, ctq^k, cuq^k; q)_\infty} \\
 &\quad \times \sum_{l, j=0}^\infty \frac{(aq^k, csq^k; q)_{j+l} (v/t; q)_j (ctq^k; q)_l}{(absq^k, cvq^k; q)_{j+l}} \frac{(bt)^j}{(q; q)_j} \frac{(bu)^l}{(q; q)_l} \\
 &= \frac{(abs, cv, cw; q)_\infty}{(bs, cs, ct, cu, cz; q)_\infty} \\
 &\quad \times \sum_{j, k, l=0}^\infty \frac{(a, cs; q)_{j+k+l} (ct; q)_{k+l} (cu, w/z; q)_k (v/t; q)_j}{(abs, cv; q)_{j+k+l} (cw; q)_k} \\
 &\quad \times \frac{(bt)^j}{(q; q)_j} \frac{(bz)^k}{(q; q)_k} \frac{(bu)^l}{(q; q)_l}.
 \end{aligned}$$

□

Setting $w = z = 0$ in (2.6), we get (2.5). Setting $v = w = z = 0$ and replace s with t in (2.6), we get (2.4) and setting $v = w = u = z = 0$ in (2.6), we get (1.22).

3 The basic identities for $h_n(x, y|q)$ and their extensions

In this section, we easily represent the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ in the form (1.25) by the Cauchy operator as follows:

$$T(y, 1; D_q)\{x^n\} = h_n(x, y|q). \tag{3.1}$$

Our operator representation for $h_n(x, y|q)$ seems simpler than the one given in (1.24). So we use our operator representation to give a short and simple operator proof for the basic identities for $h_n(x, y|q)$. Also we give extensions for the basic identities.

Theorem 3.1 (The generating function for $h_n(x, y|q)$). *We have*

$$\sum_{n=0}^\infty h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty}, \quad \max\{|t|, |xt|\} < 1. \tag{3.2}$$

Proof By using (3.1) and (1.21), we get

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} T(y, 1; D_q)\{x^n\} \frac{t^n}{(q; q)_n} \\ &= T(y, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q; q)_n} \right\} \\ &= T(y, 1; D_q) \left\{ \frac{1}{(xt; q)_{\infty}} \right\} \\ &= \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}}. \end{aligned}$$

□

Theorem 3.2 (Extended generating function for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^{\infty} h_{n+k}(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(y, xt; q)_j}{(yt; q)_j} x^{k-j}, \quad (3.3)$$

where $\max\{|t|, |xt|\} < 1$.

Proof

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n+k}(x, y|q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} T(y, 1; D_q)\{x^{n+k}\} \frac{t^n}{(q; q)_n} \\ &= T(y, 1; D_q) \left\{ x^k \sum_{n=0}^{\infty} \frac{(xt)^n}{(q; q)_n} \right\} \\ &= T(y, 1; D_q) \left\{ \frac{x^k}{(xt; q)_{\infty}} \right\} \\ &= \frac{(yt; q)_{\infty}}{(t, xt; q)_{\infty}} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(y, xt; q)_j}{(yt; q)_j} x^{k-j}. \end{aligned}$$

□

Setting $k = 0$ in (3.3), we get the generating function (3.2) for the homogeneous Rogers–Szegő polynomials.

Theorem 3.3 (Mehler’s formula for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q) \frac{t^n}{(q; q)_n} = \frac{(yt, xvt; q)_{\infty}}{(t, xt, xut; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} y, xt, v/ut \\ yt, xvt \end{matrix}; q, ut \right), \quad (3.4)$$

where $\max\{|t|, |xt|, |ut|, |xut|\} < 1$.

Proof

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q)\frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} T(y, 1; D_q)\{x^n\} h_n(u, v|q)\frac{t^n}{(q; q)_n} \\ &= T(y, 1; D_q)\left\{\sum_{n=0}^{\infty} h_n(u, v|q)\frac{(xt)^n}{(q; q)_n}\right\} \\ &= T(y, 1; D_q)\left\{\frac{(xvt; q)_{\infty}}{(xt, xut; q)_{\infty}}\right\} \\ &= \frac{(yt, xvt; q)_{\infty}}{(t, xt, xut; q)_{\infty}} {}_3\phi_2\left(\begin{matrix} y, xt, v/u \\ yt, xvt \end{matrix}; q, ut\right). \end{aligned}$$

□

Setting $u = v$ in (3.4), we get the generating function (3.2) for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$ and setting $y = v = 0$ in (3.4), we get Mehler’s formula (1.7) for the classical Rogers–Szegő polynomials.

Theorem 3.4 (Extended Mehler’s formula for $h_n(x, y|q)$) *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x, y|q) h_{n+k}(u, v|q)\frac{t^n}{(q; q)_n} \\ = \frac{(uyt, vt; q)_{\infty}}{(t, ut, xut; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(v, ut; q)_{j+l} (xut; q)_j (y/x; q)_l}{(uyt, vt; q)_{j+l} (q; q)_l} u^{k-j} (xt)^l, \end{aligned} \tag{3.5}$$

where $\max\{|t|, |xt|, |ut|, |xut|\} < 1$.

Proof

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(x, y|q) h_{n+k}(u, v|q)\frac{t^n}{(q; q)_n} \\ = \sum_{n=0}^{\infty} h_n(x, y|q) T_u(v, 1; D_q)\{u^{n+k}\}\frac{t^n}{(q; q)_n} \\ = T_u(v, 1; D_q)\left\{u^k \sum_{n=0}^{\infty} h_n(x, y|q)\frac{(ut)^n}{(q; q)_n}\right\} \\ = T_u(v, 1; D_q)\left\{\frac{u^k (uyt; q)_{\infty}}{(ut, xut; q)_{\infty}}\right\} \end{aligned}$$

$$= \frac{(uyt, vt; q)_\infty}{(t, ut, xut; q)_\infty} \sum_{l=0}^\infty \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(v, ut; q)_{j+l} (xut; q)_j (y/x; q)_l}{(uyt, vt; q)_{j+l} (q; q)_l} u^{k-j} (xt)^l.$$

□

Setting $k = 0$ in (3.5), we get Mehler’s formula (3.4) for the homogeneous Rogers–Szegő polynomials.

Now we derive the Rogers formula for the homogeneous Rogers–Szegő polynomials.

Theorem 3.5 (The Rogers formula for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^\infty \sum_{m=0}^\infty h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(ys; q)_\infty}{(s, xs, xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t \right), \quad (3.6)$$

where $\max\{|t|, |s|, |xt|, |xs|\} < 1$.

Proof

$$\begin{aligned} & \sum_{n=0}^\infty \sum_{m=0}^\infty h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty T(y, 1; D_q) \{x^{n+m}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= T(y, 1; D_q) \left\{ \sum_{n=0}^\infty \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^\infty \frac{(xs)^m}{(q; q)_m} \right\} \\ &= T(y, 1; D_q) \left\{ \frac{1}{(xt, xs; q)_\infty} \right\} \\ &= \frac{(ys; q)_\infty}{(s, xs, xt; q)_\infty} {}_2\phi_1 \left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t \right). \end{aligned}$$

□

Setting $y = 0$ in (3.6), we get the Rogers formula (1.8) for the classical Rogers–Szegő polynomials.

Theorem 3.6 (Extended Rogers formula for $h_n(x, y|q)$) *We have*

$$\begin{aligned} \sum_{n=0}^\infty \sum_{m=0}^\infty h_{n+m+k}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} &= \frac{(yt; q)_\infty}{(t, xt, xs; q)_\infty} \sum_{l=0}^\infty \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \\ &\times \frac{(y, xt; q)_{j+l} (xs; q)_j}{(yt; q)_{j+l} (q; q)_l} x^{k-j} s^l, \quad (3.7) \end{aligned}$$

where $\max\{|t|, |s|, |xt|, |xs|\} < 1$.

Proof

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m+k}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T(y, 1; D_q) \{x^{n+m+k}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\
 &= T(y, 1; D_q) \left\{ x^k \sum_{n=0}^{\infty} \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(xs)^m}{(q; q)_m} \right\} \\
 &= T(y, 1; D_q) \left\{ \frac{x^k}{(xs, xt; q)_{\infty}} \right\} \\
 &= \frac{(yt; q)_{\infty}}{(t, xt, xs; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(y, xt; q)_{j+l} (xs; q)_j}{(yt; q)_{j+l} (q; q)_l} x^{k-j} s^l.
 \end{aligned}$$

□

Setting $k = 0$ in (3.7), we get the Rogers formula (3.6) for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$.

4 Some other extended identities for $h_n(x, y|q)$

By using the Cauchy operator $T(a, b; D_q)$, we give some other extended identities for the homogeneous Rogers–Szegő polynomials $h_n(x, y|q)$.

By using identity (2.3), we give the following extended identity for $h_n(x, y|q)$.

Theorem 4.1 *We have*

$$\begin{aligned}
 & \sum_{k=0}^{\infty} h_{m+k}(x, y|q) h_{n+k}(u, v|q) \frac{t^k}{(q; q)_k} \\
 &= \frac{(vt, uyt; q)_{\infty}}{(t, ut, uxt; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^m \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \frac{(v, ut; q)_{i+l} (uxt; q)_{i+j} (y/x; q)_l (y; q)_j}{(uyt; q)_{i+j+l} (vt; q)_{i+l} (q; q)_l} \\
 & \quad \times x^{l+m-j} u^{n-i} (tq^j)^l, \tag{4.1}
 \end{aligned}$$

where $\max\{|t|, |xt|, |ut|, |xut|\} < 1$.

Proof

$$\begin{aligned}
 & \sum_{k=0}^{\infty} h_{m+k}(x, y|q) h_{n+k}(u, v|q) \frac{t^k}{(q; q)_k} \\
 &= T_u(v, 1; D_q) \left\{ u^n \sum_{k=0}^{\infty} h_{m+k}(x, y|q) \frac{(ut)^k}{(q; q)_k} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= T_u(v, 1; D_q) \left\{ u^n \frac{(uyt; q)_\infty}{(ut, uxt; q)_\infty} \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} \frac{(y, uxt; q)_j}{(uyt; q)_j} x^{m-j} \right\} \\
 &= \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} (y; q)_j x^{m-j} T_u(v, 1; D_q) \left\{ u^n \frac{(uytq^j; q)_\infty}{(ut, uxtq^j; q)_\infty} \right\} \\
 &= \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix} (y; q)_j x^{m-j} \frac{(vt, uytq^j; q)_\infty}{(t, ut, uxtq^j; q)_\infty} \sum_{l=0}^\infty \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} \\
 &\quad \times \frac{(v, ut; q)_{i+l} (uxtq^j; q)_i (y/x; q)_l}{(vt, uytq^j; q)_{i+l} (q; q)_l} u^{n-i} (xtq^j)^l \\
 &= \frac{(vt, uyt; q)_\infty}{(t, ut, uxt; q)_\infty} \sum_{l=0}^\infty \sum_{i=0}^n \sum_{j=0}^m \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} \\
 &\quad \times \frac{(v, ut; q)_{i+l} (uxt; q)_{i+j} (y/x; q)_l (y; q)_j x^{l+m-j} u^{n-i} (tq^j)^l}{(uyt; q)_{i+j+l} (vt; q)_{i+l} (q; q)_l}.
 \end{aligned}$$

□

Setting $n = m = 0$ in (4.1), we get Mehler’s formula (3.4) for $h_n(x, y|q)$. Setting $m = 0$, we get the extended Mehler’s formula (3.5) for $h_n(x, y|q)$.

By using (2.4), we give the following identity for $h_n(x, y|q)$:

Theorem 4.2 *We have*

$$\begin{aligned}
 &\sum_{n,m,k=0}^\infty h_{n+m+k}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k} \\
 &= \frac{(yt; q)_\infty}{(t, xt, xs, xv; q)_\infty} \sum_{i,j=0}^\infty \frac{(y, xt; q)_{i+j} (xs; q)_i}{(yt; q)_{i+j}} \frac{v^i}{(q; q)_i} \frac{s^j}{(q; q)_j}, \tag{4.2}
 \end{aligned}$$

where $\max\{|s|, |t|, |xs|, |xt|, |xv|\} < 1$.

Proof

$$\begin{aligned}
 &\sum_{n,m,k=0}^\infty h_{n+m+k}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k} \\
 &= \sum_{n,m,k=0}^\infty T(y, 1; D_q) \{x^{n+m+k}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k} \\
 &= T(y, 1; D_q) \left\{ \sum_{n=0}^\infty \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^\infty \frac{(xs)^m}{(q; q)_m} \sum_{k=0}^\infty \frac{(xv)^k}{(q; q)_k} \right\} \\
 &= T(y, 1; D_q) \left\{ \frac{1}{(xt, xs, xv; q)_\infty} \right\}
 \end{aligned}$$

$$= \frac{(yt; q)_\infty}{(t, xt, xs, xv; q)_\infty} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(y, xt; q)_{i+j} (xs; q)_i}{(yt; q)_{i+j}} \frac{v^i}{(q; q)_i} \frac{s^j}{(q; q)_j}.$$

□

Setting $v = 0$ in (4.2), we get the Rogers formula (3.6) for $h_n(x, y|q)$.

The following identity for $h_n(x, y|q)$ will be derived by using (2.5) as follows:

Theorem 4.3 *We have*

$$\begin{aligned} & \sum_{m=0}^\infty \sum_{k=0}^\infty h_{m+k}(x, y|q) h_{n+k}(u, v|q) \frac{t^m}{(q; q)_m} \frac{s^k}{(q; q)_k} \\ &= \frac{(ys, xvs; q)_\infty}{(s, xs, xt, xus; q)_\infty} \sum_{l,j=0}^\infty \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(y, xs; q)_{j+l} (xus; q)_{i+l} (v/u; q)_j (v; q)_i}{(ys; q)_{j+l} (xvs; q)_{i+j+l} (q; q)_j (q; q)_l} \\ & \quad \times u^{j+n-i} (sq^i)^j t^l, \end{aligned} \tag{4.3}$$

where $\max\{|s|, |xs|, |us|, |xt|, |xus|\} < 1$.

Proof

$$\begin{aligned} & \sum_{m,k=0}^\infty h_{m+k}(x, y|q) h_{n+k}(u, v|q) \frac{t^m}{(q; q)_m} \frac{s^k}{(q; q)_k} \\ &= T_x(y, 1; D_q) \left\{ \sum_{m=0}^\infty \frac{(xt)^m}{(q; q)_m} \sum_{k=0}^\infty h_{n+k}(u, v|q) \frac{(xs)^k}{(q; q)_k} \right\} \\ &= \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} (v; q)_i u^{n-i} T_x(y, 1; D_q) \left\{ \frac{(xvsq^i; q)_\infty}{(xs, xusq^i, xt; q)_\infty} \right\} \\ &= \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} (v; q)_i u^{n-i} \frac{(ys, xvsq^i; q)_\infty}{(s, xs, xusq^i, xt; q)_\infty} \\ & \quad \times \sum_{l,j=0}^\infty \frac{(y, xs; q)_{j+l} (v/u; q)_j (xusq^i; q)_l t^l (usq^i)^j}{(ys, xvsq^i; q)_{j+l} (q; q)_l (q; q)_j} \\ &= \frac{(ys, xvs; q)_\infty}{(s, xs, xt, xus; q)_\infty} \\ & \quad \times \sum_{l,j=0}^\infty \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} \frac{(y, xs; q)_{j+l} (xus; q)_{i+l} (v/u; q)_j (v; q)_i}{(ys; q)_{j+l} (xvs; q)_{i+j+l} (q; q)_j (q; q)_l} \\ & \quad \times u^{j+n-i} (sq^i)^j t^l. \end{aligned}$$

□

Setting $n = t = 0$ in (4.3), we get Mehler’s formula (3.4) for $h_n(x, y|q)$.

Theorem 4.4 *We have*

$$\begin{aligned} & \sum_{n,m=0}^{\infty} h_{n+m}(x, y|q) h_n(u, v|q) h_m(z, w|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \frac{(yt, xvt, xws; q)_{\infty}}{(t, xt, xut, xs, xzs; q)_{\infty}} \\ & \times \sum_{j,k,l=0}^{\infty} \frac{(y, xt; q)_{j+k+l} (xut; q)_{k+l} (xs, w/z; q)_k (v/u; q)_j (ut)^j (zs)^k s^l}{(yt, xvt; q)_{j+k+l} (xws; q)_k (q; q)_j (q; q)_k (q; q)_l}, \end{aligned} \tag{4.4}$$

where $\max\{|t|, |xt|, |ut|, |xut|, |xs|, |xzs|\} < 1$.

Proof

$$\begin{aligned} & \sum_{n,m=0}^{\infty} h_{n+m}(x, y|q) h_n(u, v|q) h_m(z, w|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= T_x(y, 1; D_q) \left\{ \sum_{n=0}^{\infty} h_n(u, v|q) \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^{\infty} h_m(z, w|q) \frac{(xs)^m}{(q; q)_m} \right\} \\ &= T_x(y, 1; D_q) \left\{ \frac{(xvt, xws; q)_{\infty}}{(xt, xut, xs, xzs; q)_{\infty}} \right\} \\ &= \frac{(yt, xvt, xws; q)_{\infty}}{(t, xt, xut, xs, xzs; q)_{\infty}} \\ & \times \sum_{j,k,l=0}^{\infty} \frac{(y, xt; q)_{j+k+l} (xut; q)_{k+l} (xs, w/z; q)_k (v/u; q)_j (ut)^j (zs)^k s^l}{(yt, xvt; q)_{j+k+l} (xws; q)_k (q; q)_j (q; q)_k (q; q)_l}. \end{aligned}$$

□

Setting $s = 0$ in (4.4), we get Mehler’s formula (3.4) for $h_n(x, y|q)$.

Finally, we derive the following identity for $h_n(x, y|q)$ by using (1.22) and (2.5):

Theorem 4.5 *We have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} h_{n+k}(x, y|q) h_{m+k}(u, v|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{w^k}{(q; q)_k} \\ &= \frac{(yw, xvw; q)_{\infty}}{(w, us, xt, xw, xuw; q)_{\infty}} \\ & \times \sum_{i,j,l=0}^{\infty} \frac{(y, xw; q)_{i+l} (xuw; q)_{i+j} (v; q)_j (v/u; q)_l}{(yw; q)_{i+l} (xvw; q)_{i+j+l} (q; q)_i (q; q)_j (q; q)_l} t^i s^j (uwq^j)^l, \end{aligned} \tag{4.5}$$

where $\max\{|s|, |w|, |us|, |xw|, |uw|, |xt|, |xuw|\} < 1$.

Proof

$$\begin{aligned}
& \sum_{n,m,k=0}^{\infty} h_{n+k}(x, y|q) h_{m+k}(u, v|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{w^k}{(q; q)_k} \\
&= \sum_{n,m,k=0}^{\infty} T_x(y, 1; D_q)\{x^{n+k}\} T_u(v, 1; D_q)\{u^{m+k}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{w^k}{(q; q)_k} \\
&= T_x(y, 1; D_q) T_u(v, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(us)^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{(xuw)^k}{(q; q)_k} \right\} \\
&= T_x(y, 1; D_q) \left\{ \frac{1}{(xt; q)_{\infty}} T_u(v, 1; D_q) \left\{ \frac{1}{(us, u x w; q)_{\infty}} \right\} \right\} \\
&= T_x(y, 1; D_q) \left\{ \frac{1}{(xt; q)_{\infty}} \frac{(xvw; q)_{\infty}}{(xw, us, xuw; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} v, xuw \\ xv w \end{matrix}; q, s \right) \right\} \\
&= \frac{1}{(us; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(v; q)_j s^j}{(q; q)_j} T_x(y, 1; D_q) \left\{ \frac{(xvwq^j; q)_{\infty}}{(xw, xuwq^j, xt; q)_{\infty}} \right\}.
\end{aligned}$$

Applying (2.5), we get the desired identity. \square

Setting $t = s = 0$ in (4.5), we get Mehler's formula (3.4) for $h_n(x, y|q)$.

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