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New Forms of the Cauchy Operator and Some of Their Applications

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Abstract. In this paper, we first construct the Cauchy q-shift operator $T(a, b; D_{xy})$ and the Cauchy q-difference operator $L(a, b; \theta_{xy})$. We then apply these operators in order to represent and investigate some new families of q-polynomials which are defined in this paper. We derive some q-identities such as generating functions, symmetry properties and Rogers-type formulas for these q-polynomials. We also give an application for the q-exponential operator $R(bD_q)$.

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1. INTRODUCTION AND NOTATION

We begin our investigation by reviewing some common notation and terminology for basic (or q-) hypergeometric series in (for example) [15, 22] (see also [23, 24]). We assume that the parameter q is a fixed nonzero real or complex number and $|q| < 1$. The q-shifted factorial is defined for any real or complex parameter a by

$$(\lambda; q)_0 = 1 \qquad \text{and} \qquad (\lambda; q)_n = (1 - \lambda q)(1 - \lambda q^2) \cdots (1 - \lambda q^{n-1}) \qquad (n \in \mathbb{N}) \tag{1.1}$$

and

$$(\lambda; q)_\infty = \prod_{k=0}^{\infty} (1 - \lambda q^k),$$

where $\mathbb{N}$ denotes the set of positive integers,

$$(\lambda; q)_n = \frac{(\lambda; q)_\infty}{(\lambda q^n; q)_\infty} \qquad \text{and} \qquad (\lambda; q)_{n+k} = (\lambda; q)_k \, (\lambda q^k; q)_n. \tag{1.2}$$

We also adopt the following notation for the products of several q-shifted factorials:

$$(\lambda_1, \lambda_2, \ldots, \lambda_m; q)_n = (\lambda_1; q)_n (\lambda_2; q)_n \cdots (\lambda_m; q)_n$$

and

$$(\lambda_1, \lambda_2, \ldots, \lambda_m; q)_\infty = (\lambda_1; q)_\infty (\lambda_2; q)_\infty \cdots (\lambda_m; q)_\infty.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix} \qquad (0 \leqq k \leqq n).$$

The basic (or q-) hypergeometric series ${}_r\Phi_s$ is defined by (see, for example, [22, p. 347 *et seq.*]

$${}_r\Phi_s \left[\begin{matrix} a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{matrix} \; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n, \tag{1.3}$$

provided that the series converges.

Basic (or q-) hypergeometric series and various associated families of q-polynomials are useful in a wide variety of fields including, for example, the theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, non-linear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology, and statistics (see [22, pp. 346–351] and the references cited therein).

Recently, by comparing two known q-series identities, Abdlhusein [1] derived the following important q-hypergeometric transformation between ${}_1\Phi_1$ and ${}_2\Phi_1$ [see also Eq. (2.7) below]:

$$ {}_1\Phi_1\left[\begin{matrix} xt; \\ yt; \end{matrix}\; q, ys\right] = \frac{(xt, ys; q)_\infty}{(yt; q)_\infty}\; {}_2\Phi_1\left[\begin{matrix} y/x, 0; \\ ys; \end{matrix}\; q, xt\right] \qquad (|xt| < 1). \tag{1.4} $$

An analogous q-hypergeometric transformation between ${}_1\Phi_1$ and ${}_2\Phi_1$ can be derived by applying the following known results (see, for example, [22, p. 348]):

$$ {}_2\Phi_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; q, z\right] = \frac{(c/b; q)_\infty (bz; q)_\infty}{(c; q)_\infty (z; q)_\infty}\; {}_2\Phi_1\left[\begin{matrix} abz/c, b; \\ bz; \end{matrix}\; q, c/b\right] \tag{1.5} $$

and

$$ {}_2\Phi_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; q, z\right] = \frac{(az; q)_\infty}{(z; q)_\infty}\; {}_2\Phi_2\left[\begin{matrix} a, c/b; \\ c, az; \end{matrix}\; q, bz\right]. \tag{1.6} $$

Indeed, if we first apply (1.5) to the left-hand side of (1.6) and then set $a = 0$ in the resulting equation, we readily find that

$$ {}_1\Phi_1\left[\begin{matrix} c/b\,; \\ c\,; \end{matrix}\; q, bz\right] = \frac{(c/b; q)_\infty (bz; q)_\infty}{(c; q)_\infty}\; {}_2\Phi_1\left[\begin{matrix} b, 0; \\ bz; \end{matrix}\; q, bz\right], $$

which, upon first setting $c = abz$ and then letting $a = x$, $b = y$ and $z = t$, yields the following analogue of the q-hypergeometric transformation (1.4):

$$ {}_1\Phi_1\left[\begin{matrix} xt; \\ xyt; \end{matrix}\; q, yt\right] = \frac{(xt; q)_\infty (yt; q)_\infty}{(xyt; q)_\infty}\; {}_2\Phi_1\left[\begin{matrix} y, 0; \\ yt; \end{matrix}\; q, yt\right]. \tag{1.7} $$

The Cauchy identity is given by

$$ \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k}\, x^k = \frac{(ax; q)_\infty}{(x; q)_\infty} \qquad (|x| < 1). \tag{1.8} $$

Putting $a = 0$, (1.8) becomes Euler's identity:

$$ \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k} = \frac{1}{(x; q)_\infty} \qquad (|x| < 1). \tag{1.9} $$

The inverse of Euler's identity (1.9) is given by

$$ \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}}\, \frac{x^k}{(q; q)_k} = (x; q)_\infty. \tag{1.10} $$

The Cauchy polynomials $p_n(x,y)$ are defined by

$$p_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\, x^n; \tag{1.11}$$

these polynomials satisfy the following generating function [9]:

$$\sum_{n=0}^{\infty} p_n(x,y)\,\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty} \qquad (|xt|<1), \tag{1.12}$$

where (see [9])

$$p_n(x,y) = (-1)^n q^{\binom{n}{2}} p_n(y, q^{1-n}x). \tag{1.13}$$

The generalized Rogers–Szegő polynomials $r_n(x,y)$ are defined as follows (see [14, 17]):

$$r_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k\, y^{n-k} = T(yD_q)\{x^n\}, \tag{1.14}$$

where the q-exponential operator $T(\lambda D_q)$ is defined by (see [11])

$$T(\lambda D_q) := \sum_{k=0}^{\infty} \frac{(\lambda D_q)^k}{(q;q)_k)}. \tag{1.15}$$

The bivariate Rogers–Szegő polynomials $h_n(x,y|q)$ are defined by (see [18])

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y;q)_k\, x^{n-k}. \tag{1.16}$$

The generating function and Rogers-type formula for the bivariate Rogers–Szegő polynomials $h_n(x,y|q)$ are given as follows (see [1, 9, 18]):

$$\sum_{n=0}^{\infty} h_n(x,y|q)\,\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty} \qquad (\max\{|t|,|xt|\}<1) \tag{1.17}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{n+m}(x,y|q)\,\frac{s^m}{(q;q)_m}\,\frac{t^n}{(q;q)_n} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\Phi_1\left(\begin{matrix} y, xs; \\ ys; \end{matrix}\; q,t\right) \tag{1.18}$$

$$(\max\{|s|,|t|,|xs|,|xt|\}<1).$$

The main object of this paper is first to construct the Cauchy q-shift operator $T(a,b;D_{xy})$ and the Cauchy q-difference operator $L(a,b;\theta_{xy})$. These operators are then applied in order to represent and investigate some new families of q-polynomials which are defined in this paper. We derive some q-identities such as generating functions, symmetry properties and Rogers-type formulas for these q-polynomials. We also give an application for the q-exponential operator $R(bD_q)$. Some closely-related earlier works on the general subject of our investigation include (for example) [4–6, 20, 21].

2. AN APPLICATION OF THE q-EXPONENTIAL OPERATOR $R(bD_q)$

In this section, we introduce a new q-polynomial $V_n(x, y, b|q)$ and represent it by means of the q-exponential operator $R(bD_q)$ in order to derive its generating function, symmetry property and Rogers-type formula, where the operator $R(bD_q)$ acts on the bivariate Rogers–Szegő polynomials $h_n(x, y|q)$ defined by (1.16).

Saad and Sukhi [19] introduced the following q-exponential operator:

$$R(bD_q) = \sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}} \frac{(bD_q)^k}{(q;q)_k} \tag{2.1}$$

together with its following operational rules by assuming that the operator acts on the parameter a:

$$R(bD_q)\{a^n\} = p_n(a, b), \tag{2.2}$$

$$R(bD_q)\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{(bt;q)_\infty}{(at;q)_\infty}, \tag{2.3}$$

$$R(bD_q)\left\{\frac{1}{(at, as;q)_\infty}\right\} = \frac{(bs;q)_\infty}{(at, as;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix} as; \\ bs; \end{matrix}\; q, bt\right], \tag{2.4}$$

$$R(bD_q)\left\{\frac{(av;q)_\infty}{(at, as;q)_\infty}\right\} = \frac{(bs;q)_\infty}{(as;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} v/t, b/a; \\ bs; \end{matrix}\; q, at\right]. \tag{2.5}$$

Setting $v = 0$ in (2.5), we get the following new operational rôle for the q-exponential operator:

$$R(bD_q)\left\{\frac{1}{(at, as;q)_\infty}\right\} = \frac{(bs;q)_\infty}{(as;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} b/a, 0; \\ bs; \end{matrix}\; q, at\right]. \tag{2.6}$$

By comparing our identity (2.6) and the identity (2.4), we get the following transformation:

$${}_1\Phi_1\left[\begin{matrix} as; \\ bs; \end{matrix}\; q, bt\right] = (at;q)_\infty\, {}_2\Phi_1\left[\begin{matrix} b/a, 0; \\ bs; \end{matrix}\; q, at\right], \tag{2.7}$$

which, in view of the symmetry in (2.4), is essentially the same as the q-hypergeometric transformation (1.4).

We now define a new family of q-polynomials as follows.

Definition 1. Let the q-shifted factorial and the Cauchy polynomials be defined as above. Then we define the *q-polynomials* $V_n(x, y, b|q)$ by

$$V_n(x, y, b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y;q)_{n-k}\, p_k(x, b). \tag{2.8}$$

We notice that, when $b = 0$, the definition (2.8) reduces to the definition (1.16) of the bivariate Rogers–Szegő polynomials $h_n(x, y|q)$.

The q-polynomials $V_n(x, y, b|q)$ defined by (2.8) can be represented by using the q-exponential operator $R(bD_q)$ as in Theorem 1 below.

Theorem 1. *Suppose that the operator $R(bD_q)$ acts on the variable x, Then*

$$R(bD_q)\{h_n(x, y|q)\} = V_n(x, y, b|q). \tag{2.9}$$

Proof. We observe that

$$\begin{aligned} R(bD_q)\,\{h_n(x,y|q)\} &= R(bD_q)\left\{\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(y;q)_k x^{n-k}\right\} = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(y;q)_k\,R(bD_q)\,\{x^{n-k}\} \\ &= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(y;q)_k\,p_{n-k}(x,b) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(y;q)_{n-k}\,p_k(x,b) = V_n(x,y,b|q), \end{aligned}$$

which completes the proof of Theorem 1.

The generating function of the polynomials $V_n(x,y,b|q)$ will be derived by using our representation (2.9) and the identity (2.3) of the q-exponential operator $R(bD_q)$ as follows.

Theorem 2. [Generating Function for $V_n(x,y,b|q)$]. *The q-polynomials $V_n(x,y,b|q)$ are generated by*

$$\sum_{n=0}^{\infty} V_n(x,y,b|q)\,\frac{t^n}{(q;q)_n} = \frac{(yt,bt;q)_\infty}{(t,xt;q)_\infty} \qquad (\max\{|t|,|xt|\}<1). \tag{2.10}$$

Proof. It is easily seen that

$$\begin{aligned} \sum_{n=0}^{\infty} V_n(x,y,b|q)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} R_x(bD_q)\{h_n(x,y|q)\}\frac{t^n}{(q;q)_n} = R_x(bD_q)\left\{\sum_{n=0}^{\infty} h_n(x,y|q)\frac{t^n}{(q;q)_n}\right\} \\ &= R_x(bD_q)\left\{\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\right\} = \frac{(yt;q)_\infty}{(t;q)_\infty}R_x(bD_q)\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{(yt,bt;q)_\infty}{(t,xt;q)_\infty}, \end{aligned}$$

which proves the generating function (2.10) asserted by Theorem 2.

Theorem 3. [Symmetry Property for $V_n(x,y,b|q)$]. *The following identity holds*:

$$V_n(x,y,b|q) = V_n(x,b,y|q). \tag{2.11}$$

Proof. From the generating function (2.10), we have

$$\sum_{n=0}^{\infty} V_n(x,y,b|q)\frac{t^n}{(q;q)_n} = \frac{(yt,bt;q)_\infty}{(t,xt;q)_\infty} = \frac{(bt;q)_\infty}{(t;q)_\infty}\frac{(yt;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty}\frac{(b;q)_n\,t^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(y/x;q)_k\,(xt)^k}{(q;q)_k}.$$

Now, setting $n \mapsto n-k$ and comparing the coefficients of t^n on both sides, we get the required identity.

Theorem 4. [Rogers-Type Formula for $V_n(x,y,b|q)$]. *The following double-series identity holds*:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} V_{m+n}(x,y,b|q)\,\frac{s^m}{(q;q)_m}\,\frac{t^n}{(q;q)_n} = \frac{(ys,bt;q)_\infty}{(s,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(y;q)_k\,t^k}{(ys,q;q)_k}\;{}_2\Phi_1\begin{bmatrix} b/x,0; & \\ & q,xsq^k \\ bt; & \end{bmatrix} \tag{2.12}$$

$$(\max\{|s|,|t|,|xs|,|xt|\}<1).$$

Proof. We observe that

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} V_{m+n}(x,y,b|q)\,\frac{s^m}{(q;q)_m}\,\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} R_x(bD_q)\{h_{m+n}(x,y|q)\}\frac{s^m}{(q;q)_m}\frac{t^n}{(q;q)_n} \\ &= R_x(bD_q)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{m+n}(x,y|q)\frac{s^m}{(q;q)_m}\,\frac{t^n}{(q;q)_n}\right\} = R_x(bD_q)\left\{\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\;{}_2\Phi_1\begin{pmatrix} y,xs; & \\ & q,t \\ ys; & \end{pmatrix}\right\} \\ &= R_x(bD_q)\left\{\frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(y,xs;q)_k\,t^k}{(ys,q;q)_k}\right\} = \frac{(ys;q)_\infty}{(s;q)_\infty}\sum_{k=0}^{\infty}\frac{(y;q)_k\,t^k}{(ys,q;q)_k}\,R_x(bD_q)\left\{\frac{1}{(xt,xsq^k;q)_\infty}\right\} \\ &= \frac{(ys;q)_\infty}{(s;q)_\infty}\sum_{k=0}^{\infty}\frac{(y;q)_k\,t^k}{(ys,q;q)_k}\,\frac{(bt;q)_\infty}{(xt;q)_\infty}\;{}_2\Phi_1\begin{bmatrix} b/x,0; & \\ & q,xsq^k \\ bt; & \end{bmatrix} \\ &= \frac{(ys,bt;q)_\infty}{(s,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(y;q)_k\,t^k}{(ys,q;q)_k}\;{}_2\Phi_1\begin{bmatrix} b/x,0; & \\ & q,xsq^k \\ bt; & \end{bmatrix}, \end{aligned}$$

which evidently completes the proof of Theorem 4.

3. THE CAUCHY q-SHIFT OPERATOR $T(a,b;D_{xy})$

In this section, we introduce the Cauchy q-shift operator $T(a,b;D_{xy})$. Then, by means of the operator $T(a,b;D_{xy})$, we define new q-polynomials $M_n(a,b,x,y|q)$ and represent them in terms of the Cauchy q-shift operator in order to derive their generating function. We also derive an identity for the q-exponential operator $T(bD_q)$ and use it to give the Rogers-type formula for the q-polynomials $M_n(a,b,x,y|q)$.

Chen et al. [9] introduced the homogeneous q-difference operator D_{xy} on functions in the two variables x and y, which turns out to be suitable for dealing with the bivariate Rogers–Szegő polynomials $h_n(x,y|q)$ defined by (1.16) as exhibited below:

$$D_{xy}\{f(x,y)\} = \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}, \tag{3.1}$$

where (see [9])

$$D_{xy}^k\{p_n(x,y)\} = \frac{(q;q)_n}{(q;q)_{n-k}}\,p_{n-k}(x,y) \qquad \text{and} \qquad D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k\,\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{3.2}$$

Using the q-difference operator D_{xy}, we define the Cauchy q-shift operator $T(a,b;D_{xy})$ as follows.

Definition 2. The Cauchy q-shift operator $T(a,b;D_{xy})$ is defined by

$$T(a,b;D_{xy}) = \sum_{n=0}^{\infty}\frac{(a;q)_n(bD_{xy})^n}{(q;q)_n}. \tag{3.3}$$

Theorem 5. *It is asserted that*

$$T(a,b;D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(abt,yt;q)_\infty}{(bt,xt;q)_\infty} \qquad (|bt|<1). \tag{3.4}$$

Proof. We readily see that

$$T(a,b;D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\,(bD_{xy})^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(bt)^k = \frac{(abt,yt;q)_\infty}{(bt,xt;q)_\infty},$$

which completes the proof of Theorem 5.

Definition 3. In terms of the q-shifted factorial and the Cauchy polynomials $p_n(x,y)$ defined by (1.11), we write

$$M_n(a,b,x,y|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a;q)_{n-k}\,b^{n-k}\,p_k(x,y). \tag{3.5}$$

Theorem 6. [Operator Representation for $M_n(a,b,x,y|q)$]. *The following operational formula holds*:

$$T(a,b;D_{xy})\{p_n(x,y)\} = M_n(a,b,x,y|q). \tag{3.6}$$

Proof. We have

$$\begin{aligned} T(a,b;D_{xy})\{p_n(x,y)\} &= \sum_{k=0}^{\infty}\frac{(a;q)_k(bD_{xy})^k}{(q;q)_k}\{p_n(x,y)\} \\ &= \sum_{k=0}^{n}\frac{(a;q)_k}{(q;q)_k}\,\frac{(q;q)_n}{(q;q)_{n-k}}\,b^k\,p_{n-k}(x,y) = M_n(a,b,x,y|q), \end{aligned}$$

which proves Theorem 6.

Theorem 7. [Generating Function for $M_n(a,b,x,y|q)$]. *The following generating function holds*:

$$\sum_{n=0}^{\infty} M_n(a,b,x,y|q)\,\frac{t^n}{(q;q)_n} = \frac{(abt,yt;q)_\infty}{(bt,xt;q)_\infty} \qquad (\max\{|bt|,|xt|\}<1). \tag{3.7}$$

Proof. We observe that

$$\sum_{n=0}^{\infty} M_n(a,b,x,y|q)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} T(a,b;D_{xy})\,\{p_n(x,y)\}\;\frac{t^n}{(q;q)_n} = T(a,b;D_{xy})\left\{\sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{(q;q)_n}\right\}$$
$$= T(a,b;D_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(abt,yt;q)_\infty}{(bt,xt;q)_\infty},$$

which evidently completes the proof of Theorem 7.

We now derive the identity (3.10) below for the q-exponential operator $T(bD_q)$ which will be used later in order to derive a Rogers-type formula for the q-polynomials $M_n(a,b,x,y|q)$. We first notice the following identity (see [13]):

$$D_q^k\,\{(az;q)_\infty\} = (-1)^k q^{\binom{k}{2}}\,(azq^k;q)_\infty\; z^k.$$

In this connection, we recall the following known result (see [16]):

$$T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\} = \frac{(av,absw;q)_\infty}{(as,at,aw,bs,bw;q)_\infty}\;{}_3\Phi_2\left[\begin{matrix} v/t, as, aw; \\ av, absw; \end{matrix}\; q, bt\right] \tag{3.8}$$

$$(\max\{|as|,|at|,|aw|,|bs|,|bt|,|bw|\}<1),$$

which, for $w \to 0$, yields the following identity needed in the proof of Theorem 8 below:

$$T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at;q)_\infty}\right\} = \frac{(av;q)_\infty}{(as,at,bs;q)_\infty}\;{}_2\Phi_1\left[\begin{matrix} v/t, as; \\ av; \end{matrix}\; q, bt\right] \tag{3.9}$$

$$(\max\{|as|,|at|,|bs|,|bt|\}<1).$$

Theorem 8. *The following operational formula holds*:

$$T(dD_q)\left\{\frac{(av,az;q)_\infty}{(at,aw;q)_\infty}\right\} = \frac{(av,az;q)_\infty}{(at,aw,dw;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,q^{\binom{k}{2}}\,(at,aw;q)_k\,(dz)^k}{(az,av;q)_k(q;q)_k}$$
$$\times\;{}_2\Phi_1\left[\begin{matrix} v/t, awq^k; \\ avq^k; \end{matrix}\; q, dt\right] \tag{3.10}$$

$$(\max\{|dt|,|dw|,|at|,|aw|\}<1).$$

Proof. It is easily seen that

$$
\begin{aligned}
&T(dD_q)\left\{\frac{(av,az;q)_\infty}{(at,aw;q)_\infty}\right\}\\
&=\sum_{n=0}^\infty \frac{d^n}{(q;q)_n}\, D_q^n\left\{\frac{(av,az;q)_\infty}{(at,aw;q)_\infty}\right\}\\
&=\sum_{n=0}^\infty \frac{d^n}{(q;q)_n}\sum_{k=0}^n q^{k(k-n)}\begin{bmatrix} n\\ k\end{bmatrix} D_q^k\left\{(az;q)_\infty\right\}\, D_q^{n-k}\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=\sum_{n=0}^\infty \frac{d^n}{(q;q)_n}\sum_{k=0}^n q^{k(k-n)}\begin{bmatrix} n\\ k\end{bmatrix}(-1)^k q^{\binom{k}{2}}z^k(azq^k;q)_\infty\, D_q^{n-k}\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=\sum_{k=0}^\infty\sum_{n=k}^\infty \frac{d^n}{(q;q)_k(q;q)_{n-k}}q^{k(k-n)}(-1)^k q^{\binom{k}{2}}z^k(azq^k;q)_\infty\, D_q^{n-k}\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=\sum_{n=0}^\infty\sum_{k=0}^\infty \frac{d^{n+k}}{(q;q)_n(q;q)_k}q^{-nk}(-1)^k q^{\binom{k}{2}}z^k(azq^k;q)_\infty\, D_q^{n}\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=(az;q)_\infty\sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}(dz)^k}{(az;q)_k(q;q)_k}\sum_{n=0}^\infty\frac{(dq^{-k}D_q)^n}{(q;q)_n}\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=(az;q)_\infty\sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}(dz)^k}{(az;q)_k(q;q)_k}\,T(dq^{-k}D_q)\left\{\frac{(avq^k;q)_\infty}{(atq^k,awq^k;q)_\infty}\right\}\\
&=(az;q)_\infty\sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}(dz)^k}{(az;q)_k(q;q)_k}\,\frac{(avq^k;q)_\infty}{(atq^k,awq^k,dw;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} v/t,awq^k;\\ avq^k;\end{matrix}\ q,dt\right]\\
&=\frac{(az,av;q)_\infty}{(at,aw,dw;q)_\infty}\sum_{k=0}^\infty\frac{(-1)^k q^{\binom{k}{2}}(at,aw;q)_k\,(dz)^k}{(az,av;q)_k(q;q)_k}\ {}_2\Phi_1\left[\begin{matrix} v/t,awq^k;\\ avq^k;\end{matrix}\ q,dt\right],
\end{aligned}
$$

which proves the operational formula (3.10) asserted by Theorem 8.

Theorem 9. [Rogers-Type Formula for $M_n(a,b,x,y|q)$]. *The following Rogers-type formula holds for the q-polynomials* $M_n(a,b,x,y|q)$:

$$
\begin{aligned}
&\sum_{m=0}^\infty\sum_{n=0}^\infty M_{m+n}(a,b,x,y|q)\ \frac{s^m}{(q;q)_m}\ \frac{t^n}{(q;q)_n}\\
&\qquad=\frac{(abs,ys;q)_\infty}{(bs,xs,xt;q)_\infty}\sum_{k=0}^\infty(-1)^k q^{\binom{k}{2}}\ \frac{(bs,xs;q)_k(yt)^k}{(abs,ys,q;q)_k}\ {}_2\Phi_1\left[\begin{matrix} a,xsq^k;\\ absq^k;\end{matrix}\ q,bt\right] \qquad (3.11)
\end{aligned}
$$

$$(\max\{|bs|,|bt|,|xs|,|xt|\}<1).$$

Proof.

$$
\begin{aligned}
&\sum_{m=0}^\infty\sum_{n=0}^\infty M_{m+n}(a,b,x,y|q)\frac{s^m}{(q;q)_m}\ \frac{t^n}{(q;q)_n}=\sum_{n=0}^\infty\sum_{m=n}^\infty M_m(a,b,x,y|q)\frac{t^n}{(q;q)_n}\frac{s^{m-n}}{(q;q)_{m-n}}\\
&=\sum_{n=0}^\infty\sum_{m=n}^\infty M_m(a,b,x,y|q)\,t^n\,s^{m-n}\frac{(q;q)_m}{(q;q)_m(q;q)_n(q;q)_{m-n}}=\sum_{m=0}^\infty\sum_{n=0}^m\begin{bmatrix} m\\ n\end{bmatrix}t^n\,s^{m-n}\frac{M_m(a,b,x,y|q)}{(q;q)_m}\\
&=\sum_{m=0}^\infty r_m(s,t)\,\frac{M_m(a,b,x,y|q)}{(q;q)_m}=\sum_{m=0}^\infty T(tD_q)\{s^m\}\frac{M_m(a,b,x,y|q)}{(q;q)_m} \qquad (3.12)\\
&=T(tD_q)\left\{\sum_{m=0}^\infty M_m(a,b,x,y|q)\frac{s^m}{(q;q)_m}\right\}=T(tD_q)\left\{\frac{(abs,ys;q)_\infty}{(bs,xs;q)_\infty}\right\}.
\end{aligned}
$$

The proof of assertion (3.11) of Theorem 9 will be completed when we evaluate the last expression in (3.12) by applying the q-identity (3.10) after setting $d \mapsto t$, $a \mapsto s$, $v \mapsto ab$, $z \mapsto y$, $t \mapsto b$, and $w \mapsto x$ in (3.10).

4. THE CAUCHY q-DIFFERENCE OPERATOR $L(a, b; \theta_{xy})$

In this section, we introduce the Cauchy q-difference operator $L(a, b; \theta_{xy})$ and use it to define the new q-polynomials $N_n(a, b, x, y|q)$. We then represent the q-polynomials $N_n(a, b, x, y|q)$ by means of the Cauchy q-difference operator and thereby derive their generating function. We also apply an identity involving the q-exponential operator $T(bD_q)$ to prove the Rogers-type formula for $N_n(a, b, x, y|q)$.

Saad and Sukhi [18] introduced another q-difference operator θ_{xy} for functions of two variables as follows.

Definition 4. The q-difference operator θ_{xy} is defined by

$$\theta_{xy}\{f(x,y)\} = \theta_x \eta_y D_{xy}\{f(x,y)\} = \frac{f(q^{-1}x, y) - f(x, qy)}{q^{-1}x - y}, \tag{4.1}$$

where (see [18])

$$\theta_{xy}^k \{p_n(y,x)\} = (-1)^k \frac{(q;q)_n}{(q;q)_{n-k}} p_{n-k}(y,x) \quad \text{and} \quad \theta_{xy}^k \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = (-t)^k \frac{(xt;q)_\infty}{(yt;q)_\infty}. \tag{4.2}$$

We now define the homogeneous Cauchy q-shift operator as follows.

Definition 5. The *homogeneous Cauchy q-shift operator* $L(a, b; \theta_{xy})$ is defined by

$$L(a,b;\theta_{xy}) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b\theta_{xy})^n}{(q;q)_n}. \tag{4.3}$$

Theorem 10. *The following operational formula holds*:

$$L(a,b;\theta_{xy}) \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(xt, -abt;q)_\infty}{(yt, -bt;q)_\infty} \qquad (|bt| < 1). \tag{4.4}$$

Proof. We observe that

$$\begin{aligned} L(a,b;\theta_{xy}) \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} &= \sum_{k=0}^{\infty} \frac{(a;q)_k\, b^k}{(q;q)_k} (\theta_{xy})^k \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} \\ &= \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k\, b^k}{(q;q)_k} (-t)^k = \frac{(xt, -abt;q)_\infty}{(yt, -bt;q)_\infty}, \end{aligned}$$

which prove the operational formula (4.4) asserted by Theorem 10.

Definition 6. The q-polynomials $N_n(a, b, x, y|q)$ are defined by

$$N_n(a,b,x,y|q) := \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_{n-k}\, b^{n-k}\, p_k(y,x). \tag{4.5}$$

Theorem 11. [Operator Representation for $N_n(a, b, x, y|q)$]. *It is asserted that*

$$L(a,b;\theta_{xy}) \{p_n(y,x)\} = N_n(a,b,x,y|q). \tag{4.6}$$

Proof. We observe that

$$\begin{aligned} L(a,b;\theta_{xy}) \{p_n(y,x)\} &= \sum_{k=0}^{\infty} \frac{(a;q)_k\, b^k (\theta_{xy})^k}{(q;q)_k} \{p_n(y,x)\} = \sum_{k=0}^{n} (-1)^k \frac{(a;q)_k\, b^k}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} p_{n-k}(y,x) \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k\, b^k\, p_{n-k}(y,x), \end{aligned}$$

which, upon setting $k \mapsto n - k$, yields the right-hand side of the assertion (4.6) of Theorem 11.

Theorem 12. [Generating Function for $N_n(a,b,x,y|q)$]. *The following generating function holds for the q-polynomials for $N_n(a,b,x,y|q)$:*

$$\sum_{n=0}^{\infty} N_n(a,b,x,y|q)\frac{t^n}{(q;q)_n} = \frac{(xt,-abt;q)_\infty}{(yt,-bt;q)_\infty} \qquad (\max\{|bt|,|yt|\}<1). \tag{4.7}$$

Proof. It is readily seen that

$$\sum_{n=0}^{\infty} N_n(a,b,x,y|q)\frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} L(a,b;\theta_{xy})\,\{p_n(y,x)\}\,\frac{t^n}{(q;q)_n} = L(a,b;\theta_{xy})\left\{\sum_{n=0}^{\infty} p_n(y,x)\frac{t^n}{(q;q)_n}\right\}$$
$$= L(a,b;\theta_{xy})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(xt,-abt;q)_\infty}{(yt,-bt;q)_\infty}.$$

Theorem 13. [Rogers-Type Formula for $N_n(a,b,x,y|q)$]. *The following Rogers-type formula holds for the q-polynomials $N_n(a,b,x,y|q)$:*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} N_{m+n}(a,b,x,y|q)\ \frac{s^m}{(q;q)_m}\ \frac{t^n}{(q;q)_n} = \frac{(xs,-abs;q)_\infty}{(ys,-bs,-bt;q)_\infty}$$
$$\cdot\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\ \frac{(ys,-bs;q)_k(-abt)^k}{(xs,-abs,q;q)_k}\ {}_2\Phi_1\left[\begin{matrix} x/y,-bsq^k; \\ xsq^k; \end{matrix}\ q, yt\right] \tag{4.8}$$
$$(\max\{|bs|,|bt|,|ys|,|yt|\}<1).$$

Proof. We observe that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} N_{m+n}(a,b,x,y|q)\ \frac{s^m}{(q;q)_m}\ \frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} N_m(a,b,x,y|q)\frac{t^n}{(q;q)_n}\frac{s^{m-n}}{(q;q)_{m-n}}$$
$$= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} N_m(a,b,x,y|q)\,t^n\,s^{m-n}\frac{(q;q)_m}{(q;q)_m(q;q)_n(q;q)_{m-n}} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\begin{bmatrix} m \\ n\end{bmatrix} t^n\,s^{m-n}\frac{N_m(a,b,x,y|q)}{(q;q)_m}$$
$$= \sum_{m=0}^{\infty} r_m(s,t)\,\frac{N_m(a,b,x,y|q)}{(q;q)_m} = \sum_{m=0}^{\infty} T(tD_q)\{s^m\}\frac{N_m(a,b,x,y|q)}{(q;q)_m} \tag{4.9}$$
$$= T(tD_q)\left\{\sum_{m=0}^{\infty} N_m(a,b,x,y|q)\frac{s^m}{(q;q)_m}\right\} = T(tD_q)\left\{\frac{(xs,-abs;q)_\infty}{(ys,-bs;q)_\infty}\right\}.$$

The proof of assertion (4.8) of Theorem 13 will be completed when we evaluate the last expression in (4.9) by applying the q-identity (3.10) after setting $d\mapsto t$, $a\mapsto s$, $v\mapsto x$, $z\mapsto -ab$, $t\mapsto y$, and $w\mapsto -b$ in (3.10).

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