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Pitchfork domination in graphs

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In this paper, a new model of domination in graphs called the pitchfork domination is introduced. Let $G = (V, E)$ be a finite, simple and undirected graph without isolated vertices, a subset D of V is a pitchfork dominating set if every vertex $v \in D$ dominates at least j and at most k vertices of $V - D$, where j and k are non-negative integers. The domination number of G, denotes $\gamma_{\rm pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G. In this work, pitchfork domination when $j = 1$ and $k = 2$ is studied. Some bounds on $\gamma_{\rm pf}(G)$ related to the order, size, minimum degree, maximum degree of a graph and some properties are given. Pitchfork domination is determined for some known and new modified graphs. Finally, a question has been answered and discussed that; does every finite, simple and undirected graph G without isolated vertices have a pitchfork domination or not?

Keywords: Dominating set; pitchfork domination; minimal pitchfork domination; minimum pitchfork domination.

Mathematics Subject Classification 2020: 05C69

0. Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set V and edge set E. The degree of a vertex v of any graph G is defined as the number of edges incident on v and denoted by $\deg(v)$. A vertex of degree 0 is an isolated vertex and a vertex of degree 1 is a pendant or end vertex. The vertex which is adjacent to the pendant vertex is a support vertex. The minimum and maximum degrees of vertices in G denoted by $\delta(G)$ and $\Delta(G)$, respectively. The open neighborhood of v is the set

$N(v) = \{w \in V, wu \in E\}$ and closed neighborhood is the set $N[w] = N(w) \cup \{w\}$. The subgraph of G induced by the vertices in set D is denoted by $G[D]$. The complement $\bar{G}$ of a simple graph G is the graph in which two vertices are adjacent if and only if they are not adjacent in G. For graph theoretic terminology, we refer to [6, 15, 16]. The study of domination problem is one of the fastest growing areas in graph theory. For a detailed survey of domination, one can see [7–9]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D, that is $N[D] = V$. If D has no proper subset as a dominating set then it is a minimal dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set D of G [15]. The importance of domination in various applications, led to the appearance of different types domination according to the purpose used for. The domination parameters have been formed either by imposing a condition on the induced subgraph $G[D]$ such as [1–3, 5, 14]. Or by putting a condition on the induced subgraph $G[V - D]$ in which these vertices are dominated such as [4, 12, 13, 18]. Some definition included both methods like [11, 17, 19].

Here, a new model of domination in graphs called the pitchfork domination is introduced. This type of domination is determined by the number of dominated vertices, which is beneficial for any type of networks that requires such characteristics. Some bounds on pitchfork domination number associated with order, size, minimum degree, maximum degree of a graph and some properties are given. Also, Pitchfork domination is determined for some known and new modified graphs. So that a question; does every finite, simple and undirected graph G without isolated vertices have a pitchfork domination or not? has been answered and discussed.

1. Pitchfork Domination

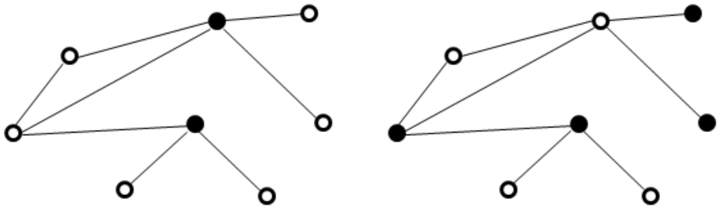
In this section, the definition for a new model of graph domination is introduced called pitchfork domination. Some theorems and properties for this type of domination are determined.

Definition 1.1. Let $G = (V, E)$ be a finite, undirected and simple graph without isolated vertices, a subset $D \subseteq V(G)$ is a pitchfork dominating set if every vertex v in D dominates only one or two vertices of $V - D$ (see Fig. 1).

Definition 1.2. A subset $D \subseteq V(G)$ is a minimal pitchfork dominating set if it has no proper pitchfork dominating set.

Definition 1.3. The minimum pitchfork dominating set D is the smallest minimal pitchfork dominating set of G. The pitchfork domination number denotes $\gamma_{\rm pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G.

Definition 1.4. Let graph G be a graph that has pitchfork domination, the minimum pitchfork dominating set of G is denoted by $\gamma_{\rm pf}$-set.



(a) Minimum dominating set (b) Minimum pitchfork dominating set

Fig. 1. The pitchfork domination.

Observation 1.5. Let G be a graph having a pitchfork domination number $\gamma_{\mathrm{pf}}(G)$, then:

(1) $|V(G)| \geq 2$.
(2) $\delta(G) \geq 1$ and $\Delta(G) \geq 1$.
(3) $\gamma_{\mathrm{pf}}(G) \geq 1$.
(4) $\gamma_{\mathrm{pf}}(G) = 1$ if and only if $G = P_2$ or P_3 or K_3.

Proposition 1.6. *Let $G = (V, E)$ be a graph having a maximum degree $\Delta(G) \leq 2$, then $\gamma(G) = \gamma_{\mathrm{pf}}(G)$.*

Proof. Let D be a minimum dominating set in G with domination number $\gamma(G)$. Since every vertex in D is adjacent to one or two vertices of $V - D$, then D is a γ_{pf}-set. □

According to the previous proposition, the following observation for the path and cycle graphs is stated.

Observation 1.7. For a path graph P_n and cycle graph C_n, we have:

(1) $\gamma_{\mathrm{pf}}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
(2) $\gamma_{\mathrm{pf}}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem 1.8. *Let D be a pitchfork dominating set of a graph G, if one of the following conditions holds then D is a minimal pitchfork dominating set*:

(1) $|N(v) \cap V - D| = 2, \forall\, v \in D$.
(2) $|N(u) \cap D| = 1, \forall\, u \in V - D$.
(3) *$G[D]$ is a null graph.*
(4) *Every vertex in D is a support vertex.*
(5) *Every vertex in D is an end vertex.*

Proof. Let D be any pitchfork dominating set in G. Suppose that D is not a minimal pitchfork dominating set, then there exists at least one vertex say $v \in D$ such that $D - \{v\}$ is a minimal pitchfork dominating set. Then we discuss the above conditions as follows:

Case 1. Suppose that the first condition holds, then there are two cases as follows.

Subcase 1. If one of the two vertices that are dominated by only vertex v. Then $D - \{v\}$ is not a pitchfork dominating set and this is a contradiction.

Subcase 2. If the two vertices which are dominated by v are also dominated by one or more vertices in $D - \{v\}$. If there is no vertex in $D - \{v\}$ that dominates v, then $D - \{v\}$ is not a pitchfork dominating set and again this is a contradiction. Otherwise, $D - \{v\}$ pitchfork dominates the vertex v by at least one vertex say y. Thus, the vertex y dominates three vertices in $V - (D - \{v\})$. Therefore, $D - \{v\}$ is not a pitchfork dominating set and this is a contradiction. In each case, $D - \{v\}$ is not a pitchfork dominating set, so, D is a minimal pitchfork dominating set.

Case 2. Suppose that the second condition holds, then for any vertex $u \in V - D$ which is dominated only by v, it is not dominated by any vertex in $D - \{v\}$. Hence, $D - \{v\}$ is not pitchfork dominating set.

Case 3. Suppose that the third condition holds, then v is not adjacent to any vertex of D since $G[D]$ is a null graph. Therefore, v is not dominated by any vertex from $D - \{v\}$. Hence, $D - \{v\}$ is not a pitchfork dominating set.

Case 4. Suppose that the fourth condition holds, then the proof is similar to proof in Case 2 where u is an end vertex.

Case 5. Suppose that the fifth condition holds, then v is not adjacent to any vertex of D since it was adjacent to only support vertex that belongs to $V - D$. Therefore, v is not dominated by any vertex from $D - \{v\}$. Hence, $D - \{v\}$ is not a pitchfork dominating set. Therefore, in all above cases, $D - \{v\}$ is not pitchfork dominating set. Hence, D is a minimal pitchfork dominating set. □

In the following theorem, the relation between the size of a graph and graph pitchfork domination number is determined.

Theorem 1.9. *Let $G = (V, E)$ be a graph of size m having a pitchfork domination number $\gamma_{\text{pf}}(G)$, then:*

$$\gamma_{\text{pf}}(G) \leq m \leq \binom{n}{2} + \gamma_{\text{pf}}^2(G) + (2 - n)\,\gamma_{\text{pf}}(G).$$

Proof. Let set D be a $\gamma_{\text{pf}}-$ set of a graph G , then:

Case 1. To prove the lower bound, suppose that $G[D]$ and $G[V - D]$ are two null graphs to be G has as few edges as possible.

Now by the definition of the pitchfork domination, there exists at least one edge from every vertex of D to $V - D$, then the number of edges from D to $V - D$ equal to $|D| = \gamma_{\text{pf}}(G)$, therefore in general $\gamma_{\text{pf}}(G) \leq m$ which is the lower bound.

Case 2. To prove the upper bound, suppose that $G[D]$ and $G[V - D]$ are two complete subgraphs to be G have maximum number of edges where the number of edges of D and $V-D$ equal to m_1 and m_2 respectively, then

$$m_1 = \frac{|D||D-1|}{2} = \frac{\gamma_{\text{pf}}(\gamma_{\text{pf}}-1)}{2},$$
$$m_2 = \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{\text{pf}})(n-\gamma_{\text{pf}}-1)}{2}.$$

Now, by the definition of pitchfork domination, there exist at most two edges from every vertex of D to $V-D$, then the number of edges from D to $V-D$ equals to $2|D| = 2\gamma_{\text{pf}}(G) = m_3$, then the number of edges of G equals to

$$\begin{aligned} m &= m_1 + m_2 + m_3 \\ &= \frac{1}{2}(\gamma_{\text{pf}}^2 - \gamma_{\text{pf}}) + \frac{1}{2}(n^2 - n\gamma_{\text{pf}} - n - n\gamma_{\text{pf}} + \gamma_{\text{pf}}^2 + \gamma_{\text{pf}}) + 2\gamma_{\text{pf}} \\ &= \gamma_{\text{pf}}^2 - n\gamma_{\text{pf}} + 2\gamma_{\text{pf}} + \frac{n^2-n}{2}. \end{aligned}$$

Which is the upper bound in general. □

Note that the lower bound of the above theorem will be sharp when $G = P_2$ where $m = \gamma_{\text{pf}}(P_2) = 1$ and the upper bound will be sharp when $G = C_3$ where $\gamma_{\text{pf}}(C_3) = 1$ and $m = 3$.

Theorem 1.10. *Let* $G = (V, E)$ *be a graph with pitchfork domination number* $\gamma_{\text{pf}}(G)$, *then*:

$$\left\lceil \frac{n}{3} \right\rceil \le \gamma_{\text{pf}}(G) \le n-1.$$

Proof. First, to prove the lower bound, let D be a γ_{pf}-set of G and $v_i, v_j \in D$ where $v_i \neq v_j$, then we have two cases.

Case 1. If $N(v_i) \cap N(v_j) \cap (V-D) = \phi$ then every vertex in $V-D$ is dominated by exactly one vertex of D. Since D is γ_{pf}-set, then every vertex in D dominates at least one vertex of $V-D$ so $\gamma_{\text{pf}}(G) = \frac{n}{2}$. And when every vertex in D dominates at most two vertices of $V-D$, then we get $\gamma_{\text{pf}}(G) = \frac{n}{3}$. Therefore, $\frac{n}{3} < \frac{n}{2} \le \gamma_{\text{pf}}(G)$.

Case 2. If $N(v_i) \cap N(v_j) \cap (V-D) \neq \phi$, then there exist one or more vertices in $V-D$ which are dominated by the two vertices v_i and v_j of D together, then $\gamma_{\text{pf}}(G) \ge \lceil \frac{n}{3} \rceil$, therefore we get the lower bound $\lceil \frac{n}{3} \rceil \le \gamma_{\text{pf}}(G)$.

The upper bound proved as follows: since every vertex in D dominates one vertex at least and two vertices at most of $V-D$, then G must contain at least one vertex in $V-D$ that is dominated by all the other $n-1$ vertices of G which will be belonging to D. Therefore, $\gamma_{\text{pf}}(G) \le n-1$. □

Note that the lower bound will be sharp when $G = P_n$ and C_n, while the upper bound will be sharp when $G = K_{1,n}$.

Corollary 1. *Let $G = (V, E)$ be a graph having a pitchfork domination number, then*:

(1) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\delta+2} \rceil$.
(2) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\Delta+2} \rceil$.
(3) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\Delta+\delta+1} \rceil$.
(4) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\delta^n+2} \rceil$.
(5) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\Delta^n+2} \rceil$.
(6) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\delta\Delta+2} \rceil$.
(7) $\gamma_{\rm pf}(G) \geq \lceil \frac{n}{\frac{\Delta}{\delta}+2} \rceil$.

Proposition 1.11. *For any graph G having a pitchfork domination set, if G has a support vertex, that is adjacent to more than two pendants then all its pendants belong to the pitchfork dominating set.*

Proof. Let D be a pitchfork dominating set of G. Suppose that v be a support vertex which is adjacent to three pendant vertices. If $v \in D$ then it dominates the three pendants which results in a contradiction with definition of pitchfork domination. Hence, $v \notin D$ and it is dominated by the pendants vertices. □

Proposition 1.12. *Let $G = (V, E)$ be a graph, then $\gamma(G) \leq \gamma_{\rm pf}(G) \leq n - 1$.*

Proof. It is clear by Theorem 1.10. □

Observation 1.13. Let G be a graph of order n and let $\deg(v) = n - 1$ for some vertices of G, then $\overline{G}$ has no pitchfork domination.

2. Pitchfork Domination of Some Families of Graphs

In this section, the pitchfork domination is determined for several known and modified families of graphs.

Proposition 2.1. *Let K_n be a complete graph with $n \geq 3$, then $\gamma_{\rm pf}(K_n) = n - 2$.*

Proof. Let D be a pitchfork dominating set in K_n. Since every vertex in D dominates at most two vertices, then $V - D$ contains only two vertices which are dominated by all the other vertices. □

Theorem 2.2. *Let H be a Hamiltonian graph of order n, then*:

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{\rm pf}(H) \leq n - 2.$$

Proof. Let $|D|$ be a pitchfork dominating set of H. Since H is a Hamiltonian graph then it contains a Hamiltonian cycle. If $\deg(v) = 2$ for all $v \in V(H)$, then H is a cycle graph and $\gamma_{\rm pf}(H) = \lceil \frac{n}{3} \rceil$ according to Observation 1.7. If $\deg(v) > 2$ for some $v \in V(H)$, then $|D|$ may be increase to avoid existence of a vertex in D that dominate more than two vertices. Hence $\gamma_{\mathrm{pf}}(C_n) \leq \gamma_{\mathrm{pf}}(H)$. If $\deg(v) = n-1$ for all $v \in V(H)$, then H is a complete graph and his pitchfork dominating set chosen according to Theorem 2.1 and $\gamma_{\mathrm{pf}}(H) = n-2$. □

Proposition 2.3. *Let G be a caterpillar graph of order $n \geq 8$ with k leaves such that $\deg(v) \geq 4$ for every non-leaf vertex, then G has a unique pitchfork dominating set D that contain only leaves with $\gamma_{\mathrm{pf}}(T) = k$.*

Proof. It is clear by Proposition 1.11. □

Theorem 2.4. *Let G be a complete bipartite graph, then:*

$$\gamma_{\mathrm{pf}}(K_{n,m}) = \begin{cases} m & \text{if } n = 2 \wedge m < 3 \quad \text{or} \quad n = 1 \wedge m > 2, \\ m-1 & \text{if } n = 2, \ m \geq 3, \\ n+m-4 & \text{if } n, \ m > 2. \end{cases}$$

Proof. Let A and B be two disjoint sets of vertices of $K_{n,m}$ such that $|A| = n$ and $|B| = m$. Three cases are obtained as follows:

Case 1. It is clear, when $n = 2 \wedge m < 3$ or $n = 1 \wedge m > 2$.

Case 2. If $n = 2$ and $m \geq 3$, suppose that $A = \{v_1, v_2\}$ and let D contain one vertex of A such as v_1 and $m-2$ vertices of B, then v_1 will dominate two vertices. Therefore, all the $m-2$ vertices of B which are in D will dominate v_2. Hence, $\gamma_{\mathrm{pf}}(K_{n,m}) = 1 + m - 2 = m - 1$.

Case 3. If $n, m > 2$, then D must be contain $n-2$ vertices of A and $m-2$ vertices of B where all the $n-2$ vertices will dominate the two vertices of B. Also, all $m-2$ vertices of B which are in D will dominate the two vertices of A that belong to $V - D$. Hence, $\gamma_{\mathrm{pf}}(K_{n,m}) = n - 2 + m - 2 = m + n - 4$.

To prove that D in the above cases is a minimum pitchfork dominating set, suppose that $\grave{D} \subseteq D$ and $|\grave{D}| < |D|$, then there exist one or more vertices of $V - D$ which are not dominated by $\grave{D}$ or there are some vertices of $\grave{D}$ dominate more than two vertices of $V - D$ which a contradiction with definition of pitchfork dominating set. Hence, D is a minimum pitchfork dominating set. □

Theorem 2.5. *Let G be a wheel graph W_n where $n \geq 3$, then:*

$$\gamma_{\mathrm{pf}}(W_n) = \begin{cases} 2\left\lceil \frac{n}{4} \right\rceil - 1 & \text{if } n \equiv 1 (\operatorname{mod} 4), \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Since wheel graph $W_n = C_n + K_1$, let us label the vertices of W_n as: $v_1, v_2, \ldots, v_{n+1}$ where $\deg(v_i) = 3$ for all $i = 1, 2, \ldots, n$ and $deg(v_{n+1}) = n$. To choose a set D, the following two cases are obtained according to n:

Case 1. If $n \equiv 0, 2, 3 \pmod 4$, let D contains two adjacent vertices from every four consecutive vertices of C_n. Hence, $D = \{u_{4i-3}, u_{4i-2}; i = 1, 2, \ldots, \lceil \frac{m}{4} \rceil\}$. Hence, D is a dominating set. Every vertex in D dominates two vertices, v_{n+1} and another vertex, except when $n \equiv 2$, there are two vertices v_1, and v_n of D dominate only v_{n+1}. Therefore, D is a $\gamma_{\rm pf}$-set and $\gamma_{\rm pf} = |D| = 2\lceil \frac{n}{4} \rceil$.

Case 2. If $n \equiv 1 \pmod 4$ let D be as in Case 1 except the last vertex will be excluded from D such that $D = \{u_{4i-3}, u_{4i-2}; i = 1, 2, \ldots, \lceil \frac{m}{4} \rceil - 1\} \cup \{v_n\}$. Where $v_n \in D$ but $v_{n-1}, v_{n-2} \notin D$. Then the vertex v_1 of D dominates only v_{n+1}, while the other vertices of D dominate v_{n+1} and another vertex. Hence, D is a $\gamma_{\rm pf}$-set and $\gamma_{\rm pf} = |D| = 2\lceil \frac{n}{4} \rceil - 1$.

To prove D in the above two cases is a minimum pitchfork dominating set, suppose that $\grave{D} \subseteq D$ and $|\grave{D}| < |D|$, then there exist at least one vertex in $V - D$ which is not dominated by any vertex of $\grave{D}$. Hence, $\grave{D}$ is not a minimum and D is the minimum. □

Observation 2.6. The triangular snake T_n of order $2n - 1$ has $\gamma_{\rm pf} = n - 1$.

The helm graph $\mathcal{H}_n$ is formed by attaching a single edge and vertex to each vertex of the cycle C_n in the wheel W_n (see [10]).

Definition 2.7. The big helm graph $\mathcal{H}_n$ is formed by attaching a single edge and vertex to each vertex of the wheel W_n, where $V(\mathcal{H}_n) = 2n+2$ and $E(\mathcal{H}_n) = 3n+1$.

Observation 2.8.

(i) For a helm graph H_n, $\gamma_{\rm pf}(H_n) = n$ for $n \geq 3$.
(ii) For a big helm graph $\mathcal{H}_n$, $\gamma_{\rm pf}(\mathcal{H}_n) = n + 1$.

The tadpole graph (dragon) $T_{m,n}$ is constructed by joining one vertex of a cycle graph C_m to a path P_n by a bridge (see [10]).

Definition 2.9. The tadpole flower graph $S_{m,n}$ is a cycle C_m in which every vertex joins with a path P_n by a bridge (see Fig. 2), where $V(S_{m,n}) = E(S_{m,n}) = m+mn$.

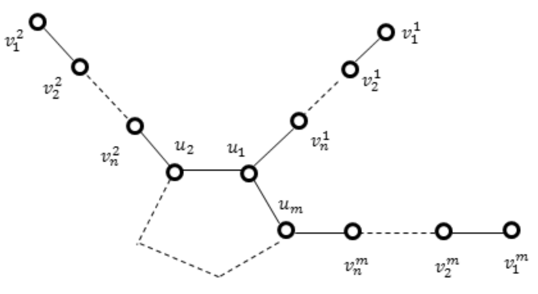


Fig. 2. The tadpole flower graph.

Theorem 2.10. *Let G be the tadpole flower graph $S_{m,n}$ where $m \geq 3$ and $n \geq 2$, then we have*:

$$\gamma_{\text{pf}}(S_{m,n}) = \begin{cases} m\left\lceil \dfrac{n}{3}\right\rceil & \textit{if } n \equiv 1,2 \pmod 3, \\ m\left\lceil \dfrac{n}{3}\right\rceil + 2\left\lceil \dfrac{m}{4}\right\rceil - 1, & \textit{if } n \equiv 0 \pmod 3 \wedge\ m \equiv 1 \pmod 4, \\ m\left\lceil \dfrac{n}{3}\right\rceil + 2\left\lceil \dfrac{m}{4}\right\rceil & \textit{if } n \equiv 0 \pmod 3 \wedge\ m \equiv 0,2,3 \pmod 4. \end{cases}$$

Proof. Let us label vertices of P_n^j, $j = 1,2,\ldots,m$ as: $\{v_i^j; i = 1,2,\ldots,n\}$ and the vertices of C_m as: $\{u_j; j = 1,2,\ldots,m\}$ where $u_j \in C_m$ is adjacent to v_n^j. Let $D = D_c \cup D_p$ where D_c is a pitchfork dominating set of C_m and $D_p = \bigcup_{j=1}^m D_j$ is a pitchfork dominating set of P_n^j, $j = 1,2,\ldots,m$. According to n, the following cases are obtained:

Case 1. If $n \equiv 1,2 \pmod 3$, then in this case, $v_n^j \in D_j$, for all $j = 1,2,\ldots,m$. Since, v_n^j is adjacent to one vertex of C_m, then v_n^j dominates one vertex of C_m. Therefore, all vertices of C_m will be dominated by D_p. Therefore, D_p is a pitchfork dominating set of $S_{m,n}$ and $\gamma_{\text{pf}}(S_{m,n}) = |D| = |D_p| = \sum_{j=1}^m |D_j| = m\,\gamma_{\text{pf}}(P_n) = m\left\lceil \frac{n}{3}\right\rceil$, where the vertices of D_j are chosen for $j = 1,2,\ldots,m$ as follows:

$$D_j = \begin{cases} \left\{v_{3i-1}^j; i = 1,2,\ldots,\left\lceil \dfrac{n}{3}\right\rceil - 1\right\} \cup \{v_n\} & \text{if } n \equiv 1, \\ \left\{v_{3i-1}^j; i = 1,2,\ldots,\left\lceil \dfrac{n}{3}\right\rceil\right\} & \text{if } n \equiv 2. \end{cases}$$

Case 2. If $n \equiv 0 \pmod 3$, then this case has two parts according to m:

Part i. If $m \equiv 1 \pmod 4$, let D_c be a set that contain two adjacent vertices from every four consecutive vertices of C_m. Then every vertex of D_c dominates one vertex of C_m and a vertex from its adjacent path which has an ordinary γ_{pf}-set with $\gamma_{\text{pf}}(P_n) = \frac{n}{3}$. Since the vertex u_{m-1} is not dominated by D_c. Therefore, it must be dominated by the last vertex of P_n^{m-1} which have $\gamma_{\text{pf}}(P_n^{m-1}) = \frac{n}{3} + 1$. Hence, D is chosen as follows:

$$D_c = \left\{u_{4i-3}, u_{4i-2}; i = 1,2\ldots,\left\lceil \frac{m}{4}\right\rceil - 1\right\} \quad \text{and}$$

$$D_j = \begin{cases} \left\{v_{3i-1}^j; i = 1,2,\ldots,\left\lceil \dfrac{n}{3}\right\rceil\right\} & \text{for } j = 1,2,\ldots,m \quad (j \neq m-1), \\ \left\{v_{3i-1}^j; i = 1,2,\ldots,\left\lceil \dfrac{n}{3}\right\rceil\right\} \cup \{v_n^j\} & \text{for } j = m-1. \end{cases}$$

Therefore, D dominates vertices of $S_{m,n}$. Since every vertex in D dominates one or two vertices, then D is a γ_{pf}-set with cardinality equal to $m\lceil\frac{n}{3}\rceil + 2\lceil\frac{m}{4}\rceil - 1$.

Part ii. If $m \equiv 0,\ 2,\ 3 \pmod 4$, then let D_c be as in part i, it contains two adjacent vertices from every four consecutive vertices of C_m. All vertices of C_m are dominated by D_c. Let D_j be the ordinary dominating set of the path, where $D_c = \{u_{4i-3}, u_{4i-2};\ i = 1, 2, \ldots, \lceil \frac{m}{4} \rceil\}$ and $D_j = \{v^j_{3i-1};\ i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$. Hence, D dominates all vertices of $S_{m,n}$ and it is a $\gamma_{\rm pf}$-set with order $m\lceil \frac{n}{3} \rceil + 2\lceil \frac{m}{4} \rceil$, since every vertex in D dominates one or two vertices.

Now, to prove D is a minimum pitchfork dominating set in all the above cases, if any vertex is deleted from D, then we get either a vertex in D dominates three vertices of $V - D$ or some vertices of $V - D$ are not dominated by any vertex of D. So, D is a minimum pitchfork dominating set. □

The lollipop graph $L_{m,n}$ is obtained by joining one vertex from complete graph K_m to a path P_n by a bridge (see [10]).

Definition 2.11. The lollipop flower $F_{m,n}$ is a complete graph K_m in which every vertex joins with a path P_n by a bridge (see Fig. 3), where $V(F_{m,n}) = m + mn$ and $E(F_{m,n}) = \binom{m}{2} + mn$.

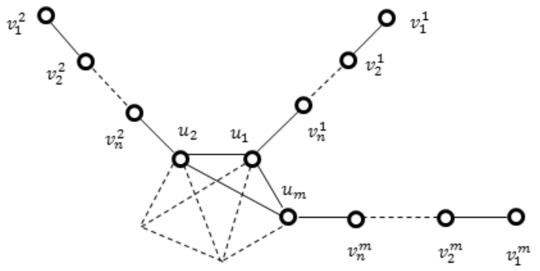


Fig. 3. The lollipop flower.

Theorem 2.12. *Let G be the lollipop flower graph $F_{m,n}$ where $m \geq 3$ and $n \geq 2$, then*:

$$\gamma_{\rm pf}(F_{m,n}) = \begin{cases} m\left\lceil \dfrac{n}{3} \right\rceil & \text{if } n \equiv 1, 2 \pmod 3 \\ m\left\lceil \dfrac{n}{3} \right\rceil + m - 1 & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

Proof. Let us label the vertices of P^i_n for $i = 1, 2, \ldots, m$ as: $\{v^i_j;\ j = 1, 2, \ldots, n\}$ and the vertices of K_m as: $\{u_i;\ i = 1, 2, \ldots, m\}$ where u_i is adjacent to v^i_n, $i = 1, 2, \ldots, m$. Let $D \subseteq V(F_{m,n})$. There are two cases as follows:

Case 1. If $n \equiv 1, 2 \pmod 3$, then let the pitchfork dominating set for every path chosen as in Theorem 2.10. Since $v^i_n \in D_i$, then v^i_n dominates u_i for all i. Hence, all vertices of K_m will be dominated by the dominating sets of the path graphs. Therefore, D is a pitchfork dominating set, and $\gamma_{\rm pf}(F_{m,n}) = m\lceil \frac{n}{3} \rceil$.

Case 2. If $n \equiv 0 \pmod 3$, according to Theorem 2.10, $v^i_n \notin D_i$ for all i and by Proposition 2.1, $\gamma_{\rm pf}(K_m) = m - 2$. Let D_K be a $\gamma_{\rm pf}$-set of K_m. Since vertices of D_k

dominate three vertices, then let $\gamma_{\rm pf}(K_m) = m-1$. So, let $(D_k) = \{u_2, u_3, \ldots, u_m\}$ then every vertex in D_k dominates u_1 and v_n^i . Hence, $D = D_i \cup D_k$, $i = 1, 2, \ldots, m$ be a pitchfork dominating set and $\gamma_{\rm pf}(F_{m,n}) = m - 1 + m \left\lceil \frac{n}{3} \right\rceil$.

Now, to prove D in the above cases is a minimum pitchfork dominating set, suppose that $\grave{D} \subseteq D$ and $|\grave{D}| < |D|$, then there exist one or more vertices in $V - D$ that are not dominated by any vertex of $\grave{D}$. Hence, $\grave{D}$ is not minimum and D is the minimum. □

The Barbell graph $B_{n,n}$, $(n \geq 3)$ is formed by connecting two copies of a complete graph K_n by a bridge (see [10]).

Definition 2.13. The corresponding Barbell graph $B^c_{n,n}$ $(n \geq 3)$ is a graph obtained by connecting two copies of complete graph K_n by a bridge between every two corresponding vertices (see Fig. 4), such that $V(B^c_{n,n}) = 2n$ and $E(B^c_{n,n}) = 2\binom{n}{2} + n$.

Proposition 2.14. *The corresponding Barbell graph* $B^c_{n,n}$ $(n \geq 3)$, *has pitchfork domination and* $\gamma_{\rm pf}(B^c_{n,n}) = 2n - 4$.

Proof. Since $\gamma_{\rm pf}(K_n) = n-2$ from Proposition 2.1 where the two adjacent vertices between the complete graphs are together either in D or not in D. □

Definition 2.15. The Barbell-path graph $Bp_{n,n,m}$, $(n, m \geq 3)$ is a graph obtained by connecting two copies of complete graph K_n by a path P_m and two bridges (see Fig. 5), where $V(Bp_{n,n,m}) = 2n + m$ and $E(Bp_{n,n,m}) = 2\binom{n}{2} + m + 1$.

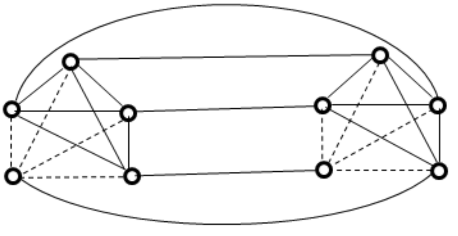


Fig. 4. The corresponding Barbell graph.

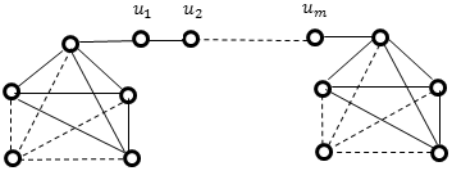


Fig. 5. The Barbell-path graph.

Proposition 2.16. *For the Barbell-path graph* $\gamma_{\text{pf}}(Bp_{n,n,m}) = 2n - 4 + \lceil \frac{m}{3} \rceil$.

Proof. Since $\gamma_{\text{pf}}(P_n) = \lceil \frac{n}{3} \rceil$ by Observation 1.7 and $\gamma_{\text{pf}}(K_n) = n - 2$ by Proposition 2.1, where the two vertices of the two copies of complete graphs which are adjacent to P_n are in $V - D$. □

Finally, every one read pitchfork domination will put the following important question:

Q: Is that every graph G has pitchfork domination or not?

Answer: No. For example, let T be a tree of order n, $n \geq 13$ having one vertex v which is adjacent to r, $r \geq 3$ support vertices v_i for $i = 1, 2, \ldots, r$. Each one of them is adjacent to k, $k \geq 3$ leaves w_j^i, $j = 1, 2, \ldots, k$. This tree has no pitchfork dominating set since all its support vertices $v_i \notin D$ and all their leaves $w_j^i \in D$ but the problem is with vertex v, where either $v \in D$ so it will dominate all the supports v_i which is contradicts our definition, or $v \notin D$ then it is adjacent to r vertices of $V - D$ and is not dominated by any vertex of D. Hence, T has no pitchfork domination.

3. Conclusion

A new type of domination "pitchfork domination" is introduced here. The relation between pitchfork domination number and the order, size, minimum degree and maximum degree is determined. The domination number can be evaluated for several standard graphs and some modified graphs formed in this paper.

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