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Total pitchfork domination and its inverse in graphs

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New two domination types are introduced in this paper. Let $G = (V, E)$ be a finite, simple, and undirected graph without isolated vertex. A dominating subset $D \subseteq V(G)$ is a total pitchfork dominating set if $1 \leq |N(u) \cap V - D| \leq 2$ for every $u \in D$ and $G[D]$ has no isolated vertex. $D^{-1} \subseteq V - D$ is an inverse total pitchfork dominating set if D^{-1} is a total pitchfork dominating set of G. The cardinality of a minimum (inverse) total pitchfork dominating set is the (inverse) total pitchfork domination number $(\gamma_{pf}^{-t}(G))$ $\gamma_{pf}^{t}(G)$. Some properties and bounds are studied associated with maximum degree, minimum degree, order, and size of the graph. These modified domination parameters are applied on some standard and complement graphs.

Keywords: Total pitchfork domination; inverse total pitchfork domination; pitchfork domination; total domination.

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1. Introduction

Let $G = (V, E)$ be a graph that has no isolated vertex with V vertex set of order n and E edge set of size m. The degree $\deg(v)$ of v in a graph G is defined as the number of edges incident with v. $\delta(G)$ and $\Delta(G)$ are the minimum degree and maximum degree, respectively in G. For graph basic concepts one can see [11]. For detailed main concepts of domination in graphs, we refer to [12, 13]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is said to be a minimal dominating set if it has no proper dominating subset.

The cardinality of minimum dominating set D of is said the domination number $\gamma(G)$. Ore [15] introduced the expression dominating set and domination number. There are more types of dominations, see [1–7, 9, 10]. Domination in graphs plays a wide role in different fields of graph theory such as labeled graph [8], topological graph [14] and fuzzy graph [16].

A new model's type of domination says the total pitchfork domination and its inverse are introduced with some bounds and properties and applied to some graphs.

2. Total Pitchfork Domination and Its Inverse

The total pitchfork domination and its inverse domination are introduced here. Some bounds and properties are studied according to the order and the size of the graph.

Definition 2.1. Let $G = (V, E)$ be a finite, simple, undirected graph, and without isolated vertices, a dominating subset D of $V(G)$ is a total pitchfork dominating set if $1 \leq |N(u) \cap V - D| \leq 2$ for every $u \in D$ and $G[D]$ has no isolated vertex. D is minimal if it has no proper total pitchfork dominating subset. D is minimum if its cardinality is smallest overall total pitchfork dominating sets, denoted by $\gamma^t_{pf}-$set. The total pitchfork domination number denoted by $\gamma^t_{pf}(G)$ is the cardinality of a $\gamma^t_{pf}-$set.

Definition 2.2. Let G be a graph with $\gamma^t_{pf}-$set D, a subset $D^{-1} \subseteq V - D$ is an inverse total pitchfork dominating set, if D^{-1} is a total pitchfork dominating set. D^{-1} is a minimal inverse total pitchfork dominating set if it has no proper inverse total pitchfork dominating subset. An inverse total pitchfork dominating set is minimum if its cardinality is smallest overall inverse total pitchfork dominating sets, denoted by $\gamma^{-t}_{pf}-$set. The inverse total pitchfork domination number denoted by $\gamma^{-t}_{pf}(G)$ is the cardinality of a $\gamma^{-t}_{pf}-$set.

Observation 2.3. Let G be a graph with $\gamma^t_{pf}-$set D, then:

(1) $\gamma^t_{pf}(G) \geq 2$.
(2) $\deg(v) \geq 2$ for every $v \in D$.
(3) Every support vertex belongs to every total pitchfork dominating set.

Observation 2.4. For any graph with an inverse total pitchfork dominating set. Then, $|V(G)| \geq 4$.

Observation 2.5. For any graph G of order n; $n = 3$. If G has total pitchfork domination, then G is a cycle.

Observation 2.6. Let G be a graph contains a pendent vertex. If G has a total pitchfork domination, then G has no inverse total pitchfork domination.

Observation 2.7. For any graph G of order n and has total pitchfork domination, if $\gamma^t_{pf}(G) > \frac{n}{2}$, then G has no inverse total pitchfork domination.

Proposition 2.8. *Let $G = (n, m)$ be a graph having total pitchfork domination, then*:

$$2 \leq \gamma^t_{pf}(G) \leq n - 1.$$

Proof. Let D be a $\gamma^t_{pf}-$ set of G, then

Case 1. Since $G[D]$ has no isolated vertex, then D has at least two vertices (adjacent together). Therefore, in general $\gamma^t_{pf}(G) = |D| \geq 2$.

Case 2. Since $V - D$ has at least one vertex such that all the other $n - 1$ vertices dominate this vertex. Hence, $\gamma^t_{pf}(G) = |D| \leq n - 1$. □

Observation 2.9. Let $G = (n, m)$ be a graph having inverse total pitchfork domination, then:

$$2 \leq \gamma^{-t}_{pf}(G) \leq n - 2.$$

Theorem 2.10. *Let $G = (n, m)$ be any graph having total pitchfork domination, then*

$$\gamma^t_{pf}(G) + \left\lceil \frac{\gamma^t_{pf}(G)}{2} \right\rceil \leq m \leq \binom{n}{2} + (\gamma^t_{pf}(G))^2 + (2 - n)\gamma^t_{pf}(G).$$

Proof. Let D be a $\gamma^t_{pf}-$ set of G, then

Case 1. Let $G[V - D]$ be a null graph, and let G have as few edges as possible. Now, by the definition of the total pitchfork domination, for every $v \in D$, then $\deg(v) = 2$ at least. Where v is adjacent with one vertex of D and dominates one vertex from $V - D$. Therefore, the number of edges between D and $V - D$ equals $m_1 = |D| = \gamma^t_{pf}(G)$. Suppose that every vertex in D adjacent with one vertex of D at least, then the number of edges of $G[D]$ equals $m_2 = \lceil \frac{|D|}{2} \rceil = \lceil \frac{\gamma^t_{pf}(G)}{2} \rceil$. Therefore, in general $m = m_1 + m_2 \geq \gamma^t_{pf}(G) + \lceil \frac{\gamma^t_{pf}(G)}{2} \rceil$.

Case 2. Suppose that $G[D]$ and $G[V - D]$ are two complete subgraphs and G having maximum number of edges. Let the number of edges of $G[D]$ and $G[V - D]$ equal m_1 and m_2, respectively, then

$$m_1 = \frac{|D||D - 1|}{2} = \frac{\gamma^t_{pf}(\gamma^t_{pf} - 1)}{2},$$

$$m_2 = \frac{|V - D||V - D - 1|}{2} = \frac{(n - \gamma^t_{pf})(n - \gamma^t_{pf} - 1)}{2}.$$

Since there exist two edges at most between every vertex of D and $V - D$, then the number of edges between D and $V - D$ equals $2|D| = 2\gamma^t_{pf}(G) = m_3$. Thus, the number of edges of G equals $m = m_1 + m_2 + m_3$. □

Theorem 2.11. *Let D be a total pitchfork dominating set of a graph G, if $|N(w) \cap D| = 1$, for all w in $V - D$, then D is a minimal total pitchfork dominating set.*

Proof. Let D be a total pitchfork dominating set in G. Assume that D is not a minimal, then there exists one vertex say $u \in D$ (at least) such that $D - \{u\}$ is a minimal total pitchfork dominating set. Thus, for any $w \in V - D$ that is dominated by u only, is not dominated by $D - \{u\}$. Thus, $D - \{u\}$ is not total pitchfork dominating set. Thus, D is a minimal total pitchfork dominating set. □

Remark 2.12. For any graph with $\gamma^t_{pf}-$set, then

(1) $\gamma(G) \leq \gamma_{pf}(G) \leq \gamma^t_{pf}(G)$.
(2) $\gamma^t(G) \leq \gamma^t_{pf}(G)$.

3. Study $\gamma^t_{pf}(G)$ and $\gamma^{-t}_{pf}(G)$ for Some Graphs

Here, the total pitchfork domination number and its inverse are applied and evaluated for some standard and complement graphs such as path, cycle, wheel, complete, complete bipartite graph and their complement.

Theorem 3.1. *Let P_n be a path; $n \geq 4$, then*

(1) *P_n has total pitchfork domination if and only if $n \neq 5, 6, 9$, where $\gamma^t_{pf}(P_n) = 2\lceil \frac{n}{4} \rceil$.*
(2) *P_n has no inverse total pitchfork domination.*

Proof. 1- Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and let D equal

$$D = \begin{cases} \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \dfrac{n}{4} - 1\right\} \\ \quad \text{if } n \equiv 0 \pmod 4, \\ \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \dfrac{n}{4} \right\rceil - 2\right\} \cup \{u_{n-2}, u_{n-1}\} \\ \quad \text{if } n \equiv 3 \pmod 4, \\ \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \dfrac{n}{4} \right\rceil - 3\right\} \cup \{u_{n-5}, u_{n-4}, u_{n-2}, u_{n-1}\} \\ \quad \text{if } n \equiv 2 \pmod 4, \\ \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \dfrac{n}{4} \right\rceil - 4\right\} \cup \{u_{n-8}, u_{n-7}, u_{n-5}, u_{n-4}, \\ \quad u_{n-2}, u_{n-1}\} \quad \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Since $\deg(u) \leq 2 \; \forall \, u \in V$ and every dominating vertex u has a neighbor in D, then u dominates only one vertex. There are four cases to prove as follows:

Case 1. If $n \equiv 0 \pmod 4$. Let us divide $V(P_n)$ into $\frac{n}{4}$ disjoint subsets, every subset contains four vertices. Let D have the second and third vertices from every subset. Then, $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \frac{n}{4} - 1\}$. Since $G[D]$ has no isolated vertex, hence, D is a $\gamma^t_{pf}-$set of P_n and $\gamma^t_{pf}(P_n) = \frac{n}{2}$.

Case 2. If $n \equiv 3 \pmod 4$. Let us divide $V(P_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 1$ subsets contains four vertices. The last one subset has only three vertices. Let D have the second and third vertices from every $\lceil \frac{n}{4} \rceil - 1$ subset, and the first and second vertices from the last one subset. Then, $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 2\} \cup \{u_{n-2}, u_{n-1}\}$. Since $G[D]$ has no isolated vertex, hence, D is a $\gamma^t_{pf}-$set of P_n and $\gamma^t_{pf}(P_n) = 2\lceil \frac{n}{4} \rceil$.

Case 3. If $n \equiv 2 \pmod 4), (n \neq 6)$. Let us divide $V(P_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 2$ subsets contains four vertices. The last two subsets have only three vertices. Let $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 3\} \cup \{u_{n-5}, u_{n-4}, u_{n-2}, u_{n-1}\}$ in the same technique of Case 2. Hence, D is a $\gamma^t_{pf}-$set of P_n and $\gamma^t_{pf}(P_n) = 2\lceil \frac{n}{4} \rceil$.

Case 4. If $n \equiv 1 \pmod 4), (n \neq 5, 9)$. Let us divide $V(P_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 3$ subsets contains four vertices. The last three subsets have only three vertices. Let $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 4\} \cup \{u_{n-8}, u_{n-7}, u_{n-5}, u_{n-4}, u_{n-2}, u_{n-1}\}$ in the same technique of case 2. Hence, D is a $\gamma^t_{pf}-$set of P_n and $\gamma^t_{pf}(P_n) = 2\lceil \frac{n}{4} \rceil$. 2- In all the above cases, P_n has no $\gamma^{-t}_{pf}-$set according to Observation 2.6 since P_n has two end vertices. □

Remark 3.2. If $n = 5, 6, 9$, then P_n has no total pitchfork domination since every dominating set in P_n has an isolated vertex.

Proposition 3.3. *A cycle C_5 does not have total pitchfork domination.*

Proof. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the set of C_5 vertices. It is clear any dominating set D has at least two adjacent vertices such as v_2, v_3. Thus, $v_1, v_4 \in V - D$. If vertex $v_5 \in V - D$, it does not dominated by D, then $v_5 \in D$ which is an isolated vertex in $G[D]$. Therefore, D is not total pitchfork dominating set and C_5 does not has total pitchfork domination. □

Theorem 3.4. *Let C_n be a cycle of order $n \geq 3; n \neq 5$, then*

(1) *C_n has total pitchfork domination, where $\gamma^t_{pf}(C_n) = 2\lceil \frac{n}{4} \rceil$.*
(2) *C_n has inverse total pitchfork domination if and only if $n \equiv 0 \pmod 4$, such that $\gamma^{-t}_{pf}(C_n) = \frac{n}{2}$. Furthermore, $D^{-1} = V - D$.*
(3) *$\gamma^t_{pf}(C_n) + \gamma^{-t}_{pf}(C_n) = n$ if and only if $n \equiv 0 \pmod 4$.*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and let D equal

$$D = \begin{cases} \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \frac{n}{4} \right\rceil - 1\right\} \\ \quad \text{if } n \equiv 0, 3 \pmod 4, \\ \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \frac{n}{4} \right\rceil - 2\right\} \cup \{u_{n-1}, u_n\} \\ \quad \text{if } n \equiv 2 \pmod 4, \\ \left\{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \left\lceil \frac{n}{4} \right\rceil - 3\right\} \cup \{u_{n-4}, u_{n-3}, u_{n-1}, u_n\} \\ \quad \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Since $\deg(u) = 2 \ \forall u \in V$ and every dominating vertex u has a neighbor in D, then u dominates only one vertex in $V - D$. So, there are four cases to prove as follows:

Case 1. If $n \equiv 0, 3 \pmod 4$. Let $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 1\}$, where D is chosen as the following two subcases:

Subcase (i). If $n \equiv 0, \pmod 4$, then D is chosen in the same technique of Theorem 3.1 Case 1.

Subcase (ii). If $n \equiv 3, \pmod 4$, Let us divide $V(C_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 1$ subsets contains four vertices. The last one subset has only three vertices. Let D have the second and third vertices from every $\lceil \frac{n}{4} \rceil - 1$ subset, and the first or second two vertices from the last one subset. Since D dominates all vertices of $V - D$ and $G[D]$ has no isolated vertex, hence, D is a $\gamma^t_{pf}-$set of C_n and $\gamma^t_{pf}(C_n) = 2\lceil \frac{n}{4} \rceil$.

Case 2. If $n \equiv 2 \pmod 4$. Let $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 2\} \cup \{u_{n-1}, u_n\}$. Let us divide $V(C_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 1$ subsets contains four vertices. The last one subset has only two vertices. Let D have the second and third vertices from every $\lceil \frac{n}{4} \rceil - 1$ subset, and the two vertices of the last one subset. Since D dominates all vertices of $V - D$ and $G[D]$ has no isolated vertex, hence, D is a γ^t_{pf}-set of C_n and $\gamma^t_{pf}(C_n) = 2\lceil \frac{n}{4} \rceil$.

Case 3. If $n \equiv 1 \pmod 4$. Let $D = \{u_{4i+2}, u_{4i+3}, ; i = 0, 1, \ldots, \lceil \frac{n}{4} \rceil - 3\} \cup \{u_{n-4}, u_{n-3}, u_{n-1}, u_n\}$. Let us divide $V(C_n)$ into $\lceil \frac{n}{4} \rceil$ disjoint subsets. Every subset of the $\lceil \frac{n}{4} \rceil - 3$ subsets contains four vertices. The last three subsets have only three vertices. Let D have the second and third vertices from every $\lceil \frac{n}{4} \rceil - 3$ subset, and the first or second two vertices of the last three subsets. Since D dominates all vertices of $V - D$ and $G[D]$ has no isolated vertex, hence, D is a $\gamma^t_{pf}-$set of C_n and $\gamma^t_{pf}(C_n) = 2\lceil \frac{n}{4} \rceil$.

Now, it is clear if $n \equiv 0 \pmod 4$, C_n has an inverse total pitchfork dominating set $D^{-1} = V - D$ its vertices are chosen in a similar way of Case 1. Otherwise, C_n has no $\gamma^{-t}_{pf}-$set since $\gamma^t_{pf}(C_n) > \frac{n}{2}$ according to Observation 2.7. □

Proposition 3.5. *Every total pitchfork dominating set in C_n is a minimal if $n = 3$ or $n \equiv 0 \pmod 4$.*

Proof. Let D be a total pitchfork dominating set of G. It is clear for $n = 3$. If $n \equiv 0 \pmod 4$, then $|N(w) \cap D| = 1 \ \forall w \in V - D$ according to Theorem 3.4 Case 1. Therefore, the result is given by Theorem 2.11. □

Theorem 3.6. *Let W_n be the wheel graph of order $(n \geq 3)$, then*

(1) *W_n has total pitchfork domination for all n, where*

$$\gamma_{pf}^{t}(W_n) = \begin{cases} 2\left\lceil \dfrac{n}{4} \right\rceil - 1 & \text{if } n \equiv 1 \pmod 4, \\ 2\left\lceil \dfrac{n}{4} \right\rceil & \text{otherwise.} \end{cases}$$

(2) *W_n has inverse total pitchfork domination if and only if $n \equiv 0 \pmod 4$ or $n = 3$, where $\gamma_{pf}^{-t}(W_n) = 2\lceil \frac{n}{4} \rceil$.*

Proof. Since the wheel graph $W_n = C_n + K_1$, let us label the vertices of W_n as: $v_1, v_2, \ldots, v_{n+1}$ where $\deg(v_i) = 3$ for all $i = 1, 2, \ldots, n$ and $\deg(v_{n+1}) = n$.

(1) There are two cases depending on n

Case 1. If $n \equiv 0, 2, 3 \pmod 4$, let D be a set containing two adjacent vertices from every four consecutive vertices of C_n. Hence, $D = \{u_{4i-3}, u_{4i-2}; i = 1, 2, \ldots, \lceil \frac{n}{4} \rceil\}$. Therefore, D is a dominating set and $G[D]$ has no isolated vertices, where every vertex in D dominates two vertices, v_{n+1} and another vertex, except when $n \equiv 2$, two vertices v_1 and v_n of D dominate the only v_{n+1}. Therefore, D is a $\gamma_{pf}^{t}-$set and $\gamma_{pf}^{t} = |D| = 2\lceil \frac{n}{4} \rceil$.

Case 2. If $n \equiv 1 \pmod 4$ let D be a set as in case 1, except the last vertex will be excluded from D such that $D = \{u_{4i-3}, u_{4i-2}; i = 1, 2, \ldots, \lceil \frac{n}{4} \rceil - 1\} \cup \{v_n\}$. Where $v_n \in D$ but $v_{n-1}, v_{n-2} \notin D$. Then, the vertex v_1 of D dominates the only v_{n+1}, while the other vertices of D dominate v_{n+1} and another vertex. Hence, D is a $\gamma_{pf}^{t}-$set and $|D| = 2\lceil \frac{n}{4} \rceil - 1$.

(2) There are three cases according to the chosen of the pitchfork dominating set D of W_n in above part as follows:

Case 1. If $n = 3$, then since D has only two consecutive vertices from C_n. Let $D^{-1} = V - D$ which contain the remaining vertex of C_n and the vertex of K_1. Then, D^{-1} is an inverse total pitchfork dominating set it has no isolated vertex and every vertex in it dominates the two consecutive vertices of D. Hence, $\gamma_{pf}^{-t}(W_3) = 2$.

Case 2. If $n \equiv 0 \pmod 4$, since D has two consecutive vertices and leave the next two consecutive vertices and so on from the cycle C_n. Let D^{-1} be a set that contain the remaining vertices of C_n. Then, every vertex of D^{-1} dominates the vertex of K_1 and another one vertex of D. Therefore, D^{-1} dominates all vertices of W_n and it is a $\gamma_{pf}^{-t}-$set. Hence, $\gamma_{pf}^{-t}(W_n) = 2\lceil \frac{n}{4} \rceil$.

Case 3. If $n \not\equiv 0 \pmod 4$ and $n \neq 3$, then the vertex of K_1 says $v_{n+1} \notin D^{-1}$ since it was dominated by more than two vertices of D. Therefore, there are three subcases as follows:

Subcase (i). If $n \equiv 1 \pmod 4$, suppose that $D^{-1} = (V - D)/\{v_{n+1}\}$ is an inverse pitchfork dominating set of W_n. Then, there exists one vertex in D is not dominated by any vertex of D^{-1} which is a contradiction. Hence, W_n has no inverse total pitchfork domination.

Subcase (ii). If $n \equiv 2 \pmod 4$, then W_n has no inverse total pitchfork domination according to Observation 2.7, since $\gamma_{pf}^t(W_n) > \frac{n+1}{2}$.

Subcase (iii). If $n \equiv 3 \pmod 4$, suppose that $D^{-1} = (V-D)/\{v_{n+1}\}$ is an inverse total pitchfork dominating set of W_n. Then, there exists one vertex in D^{-1} dominates three vertices (two from D and the vertex v_{n+1}), which is a contradiction. Hence, W_n has no inverse total pitchfork domination. □

Theorem 3.7. *Let K_n be the complete graph $(n \geq 3)$, then*

(1) *K_n has total pitchfork domination for all n and $\gamma_{pf}^t(K_n) = \gamma_{pf}(K_n) = n-2$.*
(2) *K_n has inverse total pitchfork domination if and only if $n = 4$ such that $\gamma_{pf}^{-t}(K_4) = \gamma_{pf}^{-1}(K_4) = 2$.*
(3) *$\gamma_{pf}^t(K_n) + \gamma_{pf}^{-t}(K_n) = n$ if and only if $n = 4$.*

Proof.

(1) Let D be a total pitchfork dominating set in K_n. Thus, $V-D$ must contain only two vertices.
(2) Since D contains two vertices when $n = 4$, then $D^{-1} = V-D$ is an inverse total pitchfork dominating set in K_n. If $n \geq 5$ then K_n has no inverse total pitchfork domination according to Observation 2.7 since $|D| > \frac{n}{2}$. □

Theorem 3.8. *Let $K_{n,m}$ be the complete bipartite graph, then*

(1) *$K_{1,m}$ has no total pitchfork domination for all m.*
(2) *$K_{n,m}$ has total pitchfork domination for all $n, m \geq 2$ such that*

$$\gamma_{pf}^t(K_{n,m}) = \begin{cases} 2 & \text{if } n = m = 2, \\ m-1 & \text{if } n = 2,\ m \geq 3, \\ n+m-4 & \text{if } n,\ m > 2. \end{cases}$$

(3) *$K_{n,m}$ has an inverse total pitchfork domination if and only if $n, m = 2, 3, 4$ such that $\gamma_{pf}^{-t}(K_{n,m}) = \gamma_{pf}^t(K_{n,m})$.*

Proof. Let S_1 and S_2 be two disjoint subsets of vertices of $K_{n,m}$ such that $|S_1| = n$ and $|S_2| = m$.

(1) Since every pitchfork dominating set contains either the single vertex of S_1 or vertices of S_2, where $G[S_1]$ and $G[S_2]$ are null graphs. Hence, $K_{n,m}$ has no total pitchfork domination.
(2) There are three cases:

Case 1. It is clear, when $n = m = 2$.

Case 2. If $n = 2$ and $m \geq 3$, suppose that $S_1 = \{v_1, v_2\}$ and let D contains one vertex of S_1 such as v_1 and $m - 2$ vertices of S_2. Then, v_1 dominates two vertices. Therefore, all the $m - 2$ vertices of S_2 which are in D will dominate v_2. Hence, $\gamma^t_{pf}(K_{n,m}) = m - 1$.

Case 3. If $n, m > 2$, then D must contain $n - 2$ vertices of S_1 and $m - 2$ vertices of S_2, where all the $n - 2$ vertices dominate the two vertices of S_2. Also, all $m - 2$ vertices of S_2 which are in D dominate the two vertices of S_1 that belong to $V - D$. Hence, $\gamma^t_{pf}(K_{n,m}) = m + n - 4$. 3- The proof is clear when $n, m \leq 4$. Let $m \geq 5$, for all n, since D contains two vertices of S_1 and $m - 2$ vertices of S_2 by Proof 2, then if D^{-1} contains the other two vertices of S_1, it will dominate all the $m - 2$ vertices of S_2 that belong to D, but $|D| > \frac{n+m}{2}$ which is a contradiction by Observation 2.7. Hence, $K_{n,m}$ has no inverse total pitchfork domination for $m \geq 5$. □

Theorem 3.9. *Let P_n be a path graph $(n \geq 4)$, then*

(1) $\overline{P}_n$ *has total pitchfork domination such that*

$$\gamma^t_{pf}(\overline{P}_n) = \begin{cases} 2 & \text{if } n = 4, 5, 6, \\ 3 & \text{if } n = 7, \\ n - 2 & \text{if } n \geq 8. \end{cases}$$

(2) $\overline{P}_n$ *has an inverse total pitchfork domination if and only if $n = 5, 6, 7$ such that* $\gamma^{-t}_{pf}(\overline{P}_n) = n - 3$.

(3) $\gamma^t_{pf}(\overline{P}_n) + \gamma^{-t}_{pf}(\overline{P}_n) = n$ *if and only if $n = 7$.*

Proof. (1) Since $\overline{P}_2$ is a null graph and $\overline{P}_3$ has an isolated vertex, then $\overline{P}_2$ and $\overline{P}_3$ have no total pitchfork dominating sets. Let us label $\overline{P}_n$ vertices as $\{u_i;\ i = 1, 2, \ldots, n\}$, then there are five cases as follows: If $n = 4$, let $D = \{u_1, u_4\}$ which is a unique. If $n = 5$, let D have any two adjacent vertices except $\{u_i, u_{i+2}\}$, where u_{i+1} is not dominated by u_i and u_{i+2}. If $n = 6$, let $D = \{u_2, u_5\}$. If $n = 7$, let $D = \{u_2, u_4, u_6\}$. If $n \geq 8$, then let D consist of all vertices except $\{u_1, u_n\}$. Since D is a pitchfork dominating set and $G[D]$ has no isolated vertices in all previous cases, then D is a $\gamma^t_{pf}-$set.

(2) If $n = 5$, then let D^{-1} be chosen in a similar way of D above. If $n = 6$, let $D^{-1} = \{u_3, u_4, u_6\}$. If $n = 7$, then $D^{-1} = V - D$. Since when $n = 5, 6, 7$, then D^{-1} is a pitchfork dominating set and $G[D^{-1}]$ has no isolated vertices. Thus, D^{-1} is a $\gamma^{-t}_{pf}-$set. If $n \geq 8$, then $\overline{P}_n$ has no inverse total pitchfork domination by Observation 2.7, since $\gamma^t_{pf}(\overline{P}_n) > \frac{n}{2}$.

(3) From Proofs 1 and 2, we get $D^{-1} = V - D$ just for $n = 7$. □

Theorem 3.10. *Let C_n be a cycle graph ($n \geq 4$), then*

(1) $\overline{C}_n$ *has total pitchfork domination if and only if* $n \geq 6$*. Furthermore*

$$\gamma^t_{pf}(\overline{C}_n) = \begin{cases} 2 & \text{if } n = 6, \\ n-4 & \text{if } n = 7, 8, \\ 6 & \text{if } n = 9, \\ n-2 & \text{if } n \geq 10. \end{cases}$$

(2) $\overline{C}_n$ *has inverse total pitchfork domination if and only if* $n = 6, 7, 8$ *such that*

$$\gamma^{-t}_{pf}(\overline{C}_n) = \begin{cases} 2 & \text{if } n = 6, \\ n-4 & \text{if } n = 7, 8. \end{cases}$$

(3) $\gamma^t_{pf}(\overline{C}_n) + \gamma^{-t}_{pf}(\overline{C}_n) = n$ *if and only if* $n = 8$.

Proof. *Since* $\overline{C}_3$ is a null graph. Also, $\overline{C}_4$ is a disconnected graph of two components of order two. So, there is no pitchfork dominating set D in $\overline{C}_5$ that gives $G[D]$ without isolated vertices. Then, $\overline{C}_3, \overline{C}_4$, and $\overline{C}_5$ have no total pitchfork domination.

(1) Let us label $\overline{C}_n$ vertices of as $\{v_i; i = 1, 2, \ldots, n\}$, then there are five cases: If $n = 6$, let $D = \{v_i, v_{i+3}\}$ for any integer $1 \leq i \leq 3$. If $n = 7$, let $D = \{v_i; i \text{ is odd}, i \neq 7\}$. If $n = 8$, let $D = \{v_i; i \text{ is odd}\}$. If $n = 9$, let D consist of the first two vertices from every three consecutive vertices. If $n \geq 10$, let D consist of all vertices except two vertices, but we must avoid choose $V - D = \{v_i, v_{i+2}\}$ since v_{i+1} does not dominate any vertex. In all the above cases, D is a minimum total pitchfork dominating set.
(2) According to the $\gamma^t_{pf}-$set D in Proof 1, let us choose D^{-1} as follows: If $n = 6$, then let $D^{-1} = \{v_{i+1}, v_{i+4}\}$ for any integer $1 \leq i \leq 3$. If $n = 7, 8$, then let $D^{-1} = \{v_i; i \text{ is even}\}$. Since D^{-1} is an inverse pitchfork dominating set and $G[D^{-1}]$ has no isolated vertices, then D^{-1} is a $\gamma^{-t}_{pf}-$set of $\overline{C}_n$. If $n \geq 9$, then $\overline{C}_n$ has no inverse total pitchfork domination by Observation 2.7 since $\gamma^t_{pf}(\overline{C}_n) > \frac{n}{2}$.
(3) Since $D^{-1} = V - D$ for $n = 8$. □

Theorem 3.11. *Let $K_{n,m}$ be the complete bipartite graph, then*

(1) $\overline{K}_{n,m}$ *has total pitchfork domination if and only if* $n \geq 3$*, such that*

$$\gamma^t_{pf}(\overline{K}_{n,m}) = \begin{cases} 4 & \text{for } \overline{K}_{3,3}, \\ m & \text{for } \overline{K}_{3,m},\ m \geq 4, \\ n+m-4 & \text{for } \overline{K}_{n,m},\ n, m \geq 4. \end{cases}$$

(2) $\overline{K}_{n,m}$ *has inverse total pitchfork domination if and only if* $n = m = 4$, *where* $\gamma_{pf}^{-t}(\overline{K}_{4,4}) = \gamma_{pf}^{t}(\overline{K}_{4,4}) = \gamma_{pf}^{-1}(\overline{K}_{4,4}) = 4$.
(3) $\gamma_{pf}^{t}(\overline{K}_{n,m}) + \gamma_{pf}^{-t}(\overline{K}_{n,m}) = n + m$ *if and only if* $n = m = 4$.

Proof. (1) Since $\overline{K}_{n,m}$ consists of two complete components of order n and m, respectively, then, according to Theorem 3.7, if $n = m = 3$, then D has two vertices from every component. If $n = 3$ and $m \geq 4$, then D has two vertices from the first component and $m - 2$ vertices from the second component. If $n, m \geq 4$, then D has $n - 2$ and $m - 2$ vertices. Therefore, D is a $\gamma_{pf}^{t}-$set of $\overline{K}_{n,m}$.

(2) Proofs 2 and 3 are directly given from Theorem 3.7. □

Proposition 3.12. *The complement of the complete graph* $\overline{K}_n$ *has no total pitchfork domination, so also the complement of the wheel graph* $\overline{W}_n$.

Proof. According to the definition of pitchfork domination, $\overline{K}_n$ is a null graph and $\overline{W}_n$ has an isolated vertex. □

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